Quantum cosmology at finite temperature in superstring theory

Lihui Liu

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(Quantum cosmology at finite temperature in superstring theory)

Directeur de thèse : **Hervé PARTOUCHE**

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Carlo ANGELANTONJ	Rapporteur
Massimo BIANCHI	Examinateur
Pierre BINÉTRUY	Président du jury
Emilian DUDAS	Examinateur
Costas KOUNNAS	Rapporteur
Hervé PARTOUCHE	Directeur de thèse

Summary

This thesis is dedicated to the study of cosmology induced by superstring theory at finite temperature. The thermal string scenario has the aim of establishing a unified framework for describing cosmological evolution, with gravity quantized, and matter contents derived from first principles. The cosmological solutions of string theory are determined by the low energy effective action. The latter only accommodates static solutions at tree level, while nontrivial cosmological evolution is obtained when corrections from thermal and quantum effects are taken into account. We restrict our attention to the weak coupling regime. In such cases the thermal and quantum effects back-react on the initially flat static spacetime background through an effective potential computed up to one loop level, which is a Colemann-Weinberg effective potential. It turns out that this setup describes a universe filled with an ideal string gas in quasi-static evolution, and the Colemann-Weinberg effective potential is just the Helmholtz free energy density of the string gas.

The resulting cosmological evolution can be divided into three stages characterized by the scale of temperature. They are namely: 1) the Hagedorn era where the temperature is of order string scale, and the free energy density diverges due to the exponential growth of degeneracies with mass level; 2) the standard cosmology era where the temperature goes below the electroweak phase transition scale, and the nucleosynthesis takes place giving birth to the matter contents of the current universe; 3) the intermediate era which is between the above two, where the spacetime metric evolves in the pattern of a radiation-dominated universe (radiation-like), moduli can be stabilized, and the hierarchy for supersymmetry breaking scale is generated.

The issue of moduli stabilization in the intermediate era is intensively studied. At certain points in the moduli space, extra massless states emerge, and the Helmholtz free energy density, or the effective potential, develops local minima. The latter provide moduli attractors. The depth of the local minima is time dependent, which induces scalar masses reducing with cosmological evolution. This makes the coherent scalar oscillations dilute before nucleosynthesis, and the cosmological moduli problem is avoided. Specific models are studied, where attention is given to moduli stabilization by non-perturbative effects.

We first studied cosmology induced by a maximally supersymmetric heterotic string gas. The free energy density reaches local minima where perturbative string states of nonzero winding and momentum numbers become massless, giving rise to non-Abelian gauge symmetry enhancement. This can stabilize all heterotic moduli but the dilaton, i.e. the internal metric, the internal B-field and Wilson lines, among which the internal metric components are attracted to the string scale. Through the type I/heterotic string S-duality this mechanism can be mapped to the type I side. In particular it is found that the dual type I moduli are stabilized by either non perturbative BPS D-string states or by perturbative open string states, where the internal geometric moduli are stabilized at the scale $\sqrt{\lambda_{\rm I}}$, with $\lambda_{\rm I}$ the type I string coupling in ten dimensions. Enhanced gauge symmetries at moduli attractors on the heterotic side are also sent to the type I dual side. Although these enhancements of type I gauge symmetry are non-perturbative effects, they should be treated on equal footing with the gauge group induced by perturbative states.

The second case is the cosmology induced by a gas of type II string compactified on Calabi-Yau three-folds. Moduli attractors are found to be at the loci where some 2-spheres or 3-spheres in the Calabi-Yau space shrink to zero size leading the Calabi-Yau space to a singular configuration. These can be either conifold loci or some non Abelian gauge symmetry loci. In type IIA description, in the case of shrinking 2-spheres, the extra massless states arise from BPS D2-branes wrapping these 2-cycles. In case of shrinking 3-spheres, the extra massless states are not yet identified, but their existence can be inferred from the change in moduli space dimension, and further confirmed by analyzing the low energy effective action. This mechanism can lift the whole Kähler moduli space, while in the complex structure moduli space, the flat directions lifted are those associated to the shrinking 3-spheres that can be blown up into 2-spheres. The universal hypermultiplet moduli, which contains the dilaton, cannot be lifted by this mechanism. An explicit example is analyzed where all Kähler moduli are stabilized at the intersection of a conifold locus and a non-Abelian locus. By virtue of the type II/heterotic string duality, the moduli in the dual heterotic string are stabilized, where remarkably, the axio-dilaton modulus is stabilized at order 1 in the unit of string length.

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Contents

Chapter 1

Introduction

1.1 Unification and superstring theory

A prevailing conviction about the high energy physics is the unification of the four fundamental interactions. This is not only for aesthetic reason but also for addressing fundamental problems of theoretical physics such as the determination of standard model parameters, the generation of baryon-antibaryon asymmetry. Supersymmetry provides a theoretical framework which favors the unification. Even though supersymmetry is not yet directly confirmed by experiments, it is possible however, by tuning the supersymmetry breaking scale, to give rise to the grand unification theory (GUT) [1]. That is, the renormalization group flows are such that the coupling constants of the electroweak interaction and the strong interaction converge to the same value at high energy scale.

If we admit that the GUT scenario is an intermediate step towards the "theory of everything", a theory unifying all the four interactions, then a natural way to bring gravity into play is to make the global supersymmetries local. This is because invariance under local supersymmetry transformation implies invariance under diffeomorphism transformation, and the latter is essential symmetry of general relativity. In the gauging of the global supersymmetry, spin-3/2 fields appear as the "gauge fermion" coupled to the supercurrent, making it necessary to further introduce spin-2 graviton to close the supersymmetry algebra. This leads to the supergravity theory. However the theory is plagued by non-renormalizability, when we quantize the graviton as metric fluctuations around a fixed background.

It is then realized that string theory can provide way out to the difficulty of supergravity. On one hand, superstring theories accommodate supergravity as the low energy effective theory. The quantization of superstring yields supersymmetric spectrum of spacetime states. The ground level, which is of mass zero, contains a spin-2 mode, recognized as the graviton. Therefore the supergravity action becomes the Wilsonian effective action of superstring where all massive modes are integrated out. On the other hand, the spatial dimension of strings, which delocalizes the range of their interaction, introduces a natural UV cut-off. Therefore supergravity appears non-renormalizabile just because it is the low energy effective theory of some underlying UV finite theory.

However this picture of unification is not without price to pay. Most obviously, superstring spectrum generously offers much too more spacetime fields than what the standard model needs, so that the connection to the real world is not clear. Also the spacetime has to contain extra dimensions in order for the quantized string theory to respect the symmetries that exist on the classical level. Then upon compactification down to 4D spacetime, we face with highly degenerate vacua, parameterized by a considerable amount of moduli, which are massless scalar fields. These are the main problems that have to be coped with in the phenomenological application of string theory. To make string theory phenomenologically viable is the deep-seated motivation in many of the research works on string theory, for instance compactification, model building, moduli stabilization, etc. Despite these difficulties among others, string theory is the most promising candidate quantum theory of unification leading to a sensible account of phenomenology.

1.2 Cosmology induced by string theory

As is the case for any high energy theory in physics, in order to make contact with the observable universe, it is important that string theory pass the trial of cosmological application. The widely adopted cosmological scenario is that of the standard Big-Bang cosmology or the ΛCDM model. It supposes that the universe starts off from a singular and extremely high energy event, namely the Big-Bang. Very soon after, it experiences a short inflationary period, which accounts for the flatness, isotropy and homogeneity that we observe today. After the reheating pending the inflation, the universe goes through a series of symmetry breakings as it cools, including the separation of strong interaction, supersymmetry breaking and the electroweak symmetry breaking, where the last event happens when temperature drops to about $\mathcal{O}(100)$GeV scale. At this moment Higgs potential is destabilized, triggering the electroweak phase transition, giving mass to fundamental particles and the four fundamental interactions appear as what we observe today. As the universe continue cooling, it becomes radiation-dominated, and meanwhile matter starts to form through the process of nucleosynthesis, and gradually matter dominates the universe. The theoretical matter content in today's universe is obtained by fitting the model with empirically supposed matter contents, each characterized by the state equation $P = w\,\rho$, into observation results of supernovae or galaxies. This yields the proportions that each type of matter takes up: baryonic matter ~ 3%, dark matter ~ 27%, and dark energy ~ 70%.

Although the standard cosmological model has largely promoted our understanding of current observation results of the universe, it has the drawback of being a too phenominological approach. Its matter contents are put by hand, where indeed some underlying theory is desired so that these cosmic fluids can be derived from first principle. Also the application of standard model has the difficulty from the hierarchy problem at high energy scale, where the Higgs mass is attracted to an extremely large value. Then if we resolve the hierarchy problem using supersymmetric standard model scenario, the problem is transplanted into the hierarchy of supersymmetry breaking scale. Furthermore since gravity is not quantized, the model does not propose solution to the Big-Bang singularity. Last but not least, phase transitions between different cosmological eras are imposed rather than natural.

Therefore in view of the fact that string theory unifies all interactions and matter contents in a quantum theory framework, we are tempted to formulate cosmology using string theory. It should be stressed that by doing so, we are not just putting strings into the ΛCDM universe to see how they evolve, as what we do with standard model. Rather, we are requesting the string theory to generate the whole cosmology as solution to its equations of motion derived from the effective supergravity theory. However we already meet with the difficulty that for most of the cases only static AdS or flat backgrounds can be obtained at tree level as the vacuum configuration, which is already known in supergravity.

Thus effects beyond tree level must be considered. Actually in the works prior to those to be discussed in this thesis [2–7], the thermal and quantum effects are inspected for weakly coupled strings at one loop level. It is found that in certain cases the resulting corrections are under control at full string scale. These corrections induce non trivial cosmological evolution through its back reaction on the tree level solution. We will refer to this approach the "thermal string scenario" in this thesis. The logic of setting up this scenario is as follows. The universe is described at tree level by no-scale type supergravity [8], which is characterized by vanishing scalar potential minima, as well as the spontaneous breaking of supersymmetry at the scale given by the no-scale modulus. This is to account for the observation result that the universe is almost flat and has a tiny cosmological constant. As the supersymmetry is spontaneously broken by thermal effect among other possible mechanisms, a non trivial vacuum-vacuum amplitude is generated at one-loop level. The latter corrects the tree level no-scale supergravity action as the one-loop Colemann-Weinberg effective potential, and since the string coupling is set to be weak, higher order corrections can be neglected. Provided that the one-loop amplitude at finite temperature is just the logarithmic of the canonical partition function of ideal gas, the corrected action describes in effect a universe filled with a string gas at finite temperature. From the one-loop part of the action one can derive the energy density ρ and the pressure P of the string gas which sources cosmological evolution, and these string theory quantities do not have UV ambiguity as in field theories. The solution describes a universe evolving in the pattern of a radiation-dominated universe: $a(t) \propto T(t)^{-1}$ and $H(t)^2 \propto a(t)^{-D}$, where T is the temperature, a the scale factor in the flat Robertson-Walker metric (flat because of no-scale), H the Hubble parameter and t the cosmological time. What is remarkable is that the solution of nontrivial cosmological evolution is induced by the thermal string gas and is purely a quantum effect.

For constructing realistic models supersymmetry should be spontaneously broken at zero temperature. Interesting phenomenology has been unraveled for the cases where supersymmetry breaking implemented by Scherk-Schwarz reduction in an internal circle [3–6]. The associated supersymmetry breaking scale, which are of order the inverse radius, is found to be evolving proportionally with temperature $M_{\overline{\text{SUSY}}}(t) \propto T(t)$, and ratio between the two is attracted to a fixed value of order 1. The hierarchy $M_{\overline{\text{SUSY}}} \ll M_{\text{Planck}}$ is thus dynamically generated. It should be stated that the backward extrapolation of these solutions is limited by the appearance of Hagedorn instability at ultra high temperature which is about of the string scale[1] $M_{\text{s}} = \sqrt{1/\alpha'}$. There, the one-loop correction diverges at a critical temperature called the Hagedorn temperature, where the thermal string scenario breaks down. One difficulty appears in that we cannot know what the initial conditions are at the beginning of the phase where temperature falls below the Hagedorn temperature and one-loop correction become calculable. However remarkably the radiation-like solutions described above are insensitive to the initial conditions. That is, the evolution will be dynamically attracted to the radiation-like solution whatever the initial conditions.

Another appealing thing that we expect from the string cosmology is establishing a unified description of all the cosmological eras from the very beginning to the standard matter dominated era. We have just mentioned briefly the backward extrapolation of cosmological solutions, which stops at the moment where temperature is of about the string scale. Likely, the forward extrapolation should also be limited, since with the dropping of the temperature and the supersymmetry breaking scale, we will end up with a supersymmetric vacuum. Indeed at the moment when temperature lowers to about the electroweak scale Λ_{ew}, it is expected that certain infrared effects, which was screened at hight energy scale, become relevant, and the resulting radiative corrections lead to the stabilization of $M_{\overline{\text{SUSY}}}$ at about TeV scale. The analysis of this effect in supergravity context has been carried out in Refs [9, 10].

Therefore we can divide the thermal-quantum induced cosmological evolution in three eras: the *Hagedorn era* for $T \sim M_{\text{s}}$, the *standard cosmology era* for $T > \Lambda_{\text{ew}}$, and finally the cosmological era

[1] $\alpha' = l_s^2$ is the Regge slope, l_s is the string length.

which accommodates the radiation-like solutions we have just described, satisfying $M_{\rm s} \gg T \gg \Lambda_{\rm ew}$, referred to as the *intermediate era*. While the intermediate era has been intensively studied, the other two eras are less well understood. For the Hagedorn era, some works have been done where the Hagedorn singularity is resolved in type II string by implementing "gravito-magnetic fluxes" [11–14]. It has been shown that such resolution of Hagedorn singularity can lead to the resolution of the Big-Bang singularity, where we obtain a bouncing universe [12,13] or an emerging universe [14].

1.3 Moduli stabilization

As the end of Sec.1.1 complains, compactification of string theory leads to highly degenerate vacua, characterized by parameters spanning the moduli space. These parameters, or moduli, are given by the vacuum expectation values (VEV's) of massless scalar fields in the tree level effective field theory, which we refer to as moduli fields. These massless scalar fields are undesirable for phenomenological application because if they existed in nature, they would mediate new types of interaction so that we would have more than four fundamental interactions. Moreover since moduli are free parameters that the couplings and the mass spectrum depend on, the model loses predicability. Therefore if we believe in string theory as the fundamental theory of nature, then any sensible phenomenological application must manage to let moduli be settled to some fixed value and become massive scalar fields. In other words, the model must be able to generate some nontrivial scalar potential beyond tree level that lifts the flat directions.

Actually the thermal string scenario provides such mechanism. It was shown in [15] that a gas of string modes, which carry both winding and momenta, generate a free energy that enables stabilization of radii moduli. The thermal string scenario provides a quantum version of this effect, which is effectuated through the quantum one-loop correction [16]. More accurately, the Helmholtz free energy density derived from the one-loop amplitude interferes in the effective action as a scalar potential and lifts flat directions. It reaches local minima whenever there are states in the spectrum, whose mass depend on moduli, becoming massless. Therefore these points in the moduli space are just the moduli attractors. In the results in [5, 6] the moduli are attracted to points with enhanced gauge symmetry. The effective potential develops local minima due to the extra perturbative states which become massless to supply the non Cartan components of the enhanced gauge group. In fact the link between local extrema of one-loop amplitude and enhanced gauge symmetry has been explored in [17], where it is shown that the correlation of the two is true to all loop levels.

Moduli stabilization by non-perturbative effects is considered in [18, 19], which will be given much attention in the thesis. In [18] the non-perturbative effects examined are induced by D1-branes in the type I string, where the calculation is done indirectly through the dual heterotic string. For simplicity, models considered therein are maximally supersymmetric, while generalization to non-supersymmetric vacua is obvious. On the heterotic side it is found that all moduli, except the dilaton, can be stabilized at some gauge symmetry enhancement points, where F-string states are responsible for the gauge symmetry enhancement. Using string-string S-duality, this mechanism can be mapped to the dual type I side to stabilize type I moduli. In particular, given that the heterotic F-string states are mapped to type I D-string states, the moduli stabilization on the dual type I side is a non-perturbative effect, accompanied with non-perturbative gauge symmetry enhancement.

In [19] we look into the D2(D3)-brane effects in type IIA(B) strings compactified on Calabi-Yau three-folds. The moduli attractors are expected to be located at certain loci of topology change of the Calabi-Yau space. This is motivated by the analysis in [20] which reveals that singularities appear in the low energy supergravity action when the Calabi-Yau space undergoes conifold transition. This is interpreted as arising from light D-brane states, which become massless at the transition point, wrongly integrated out from the Wilsonian effective action. The singularities are repaired as we "integrate in" these D-brane states, and it turns out that these light states are weakly coupled to the Abelian gauge group. As the CFT computation of one-loop amplitude is not available in generic Calabi-Yau compactifications, the one-loop correction can be computed perturbatively (in the sense of gauge coupling not string coupling) from this repaired effective action by field theory method. We then rely on this one-loop correction to indicate moduli attractors. Similar mechanism exists when the Calabi-Yau space undergoes the extremal transition where the singular configuration contains a curve of A_{N-1} type singularity. The same procedure as for the conifold case still applies here, except that the light non-perturbative spectrum is more complicated. They form an $SU(N)$ gauge group with matter transforming in the adjoint representation. In both cases we observe that Kähler moduli and complex structure moduli moduli are attracted to values corresponding to the singular configuration of the internal Calabi-Yau space.

A major difficulty in moduli stabilization is that cosmology imposes severe conditions on the scalar masses. Basically as the universe expands, small initial fluctuation of background scalar fields in the potential well can dominate the energy of the universe at late time. For example in 4D, the oscillations store an energy density scaling as T^{-3} with T temperature of radiation, which eventually dominates over the radiation energy which scales as T^{-4} [21]. This domination continues until the corresponding scalar particles decay. Severe problem arises because not only the productions of the decay can alter the primordial abundances of light nuclei produced by nucleosynthesis, but also the huge amount of entropy production during the decay can wash out the baryon number asymmetry. This problem is termed as the cosmological moduli problem, which was initially identified in the framework of supersymmetric standard models [22–24]. One plausible solution to these is to require the scalar masses be of $\mathcal{O}(10)$TeV order, for example in the KKLT scenario [25]. It is pointed out in [23] that once this is satisfied, the decay of these scalar particles reheats the universe to a temperature of order 1MeV, high enough to restart the nucleosynthesis. Then it is found in [24] that the baryon number asymmetry can also be saved by the $\mathcal{O}(10)$TeV order scalar mass if the baryogenesis is due to the Affleck-Dine mechanism [26].

The thermal string cosmology addresses the cosmological problem differently. This is already explored in [5, 6] where it is shown that the induced scalar mass is proportional to $T^{\frac{D}{2}-1}$, which decreases in time rather than being constant. Given that the oscillation in the potential well has the frequency proportional to the square-root of the induced mass, the decrease of mass slows down the background oscillation and hence the expansion of the universe dilutes the oscillation energy faster than for constant induced mass. By consequence, one finds that the energy stored in the background scalar oscillation never dominates so that the cosmological problem does not appear.

1.4 Organization of the thesis

The first three chapters following the introduction provide preliminary string theoretical elements that our work is based on. In Chapter 2 we go rapidly through the perturbative approaches of string theory quantization and will especially be concerned about the description of the spectrum, where lightcone gauge will be used. The aim is to get a quick access to the computation of one-loop vacuum-to-vacuum amplitudes. Chapter 3 collects the conceptual and technical aspects in compactification that will later be useful in giving rise to moduli stabilization mechanism in thermal string cosmology. These include worldsheet instanton, gauge symmetry enhancement, supersymmetry breaking by orbifold, and Calabi-Yau compactification. Chapter 4 goes one step further into the non-perturbative realm, and gives account for the non-perturbative mechanisms that have been explored in our works. The discussion will be restricted on the non-perturbative D-sting effects in type I string which can be revealed by the S-duality between type I and heterotic, as well as those in Calabi-Yau 3-fold compactification of type II strings when the Calabi-Yau space undergoes extremal transition.

The next two chapters are dedicated to building up the thermal string cosmology scenario, which summarize the foundational elements in the related works [2–7, 18, 19]. They give account to the two essential aspects concerned: string gas thermodynamics and its implementation in cosmology. Chapter 5 deals with ideal string gas without considering cosmological context. Computation of the partition function $\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H}$ is described in detail, where we establish the first quantization computation of the partition function through analogy with field theory

$$\mathcal{Z}_{\text{field}} = \exp\Big[\qquad\Big] \longrightarrow \mathcal{Z}_{\text{str}} = \exp\Big[\qquad\Big]$$

so that computing the thermal partition function attributes to computing thermal one-loop amplitude. Explicit computation is performed on specific string models. Then the investigation of general properties of the thermal one-loop amplitudes leads to the discussion of Hagedorn singularity. In Chapter 6 we set up the formalism describing cosmology. The aim is to introduce the assertion that the thermal quantum effects of the ideal string gas back reacts on the background spacetime metric and fields through a stringy version of Coleman-Weinberg effective potential. This is again inspired by the field theory analogue of 1PI effective action:

$$\Gamma_{\text{field}} = S^{\text{tree}}_{\text{field}} + S^{\text{1-loop}}_{\text{field}} + \dots \qquad \longrightarrow \qquad \Gamma_{\text{str}} = S^{\text{tree}}_{\text{str}} - \qquad + \dots .$$

Then a simple application is shown, where the cosmological solution is found in maximally supersymmetric heterotic string. Based on the solution, meanwhile supplemented with the results from previous works especially on non-supersymmetric tree level vacua, we illustrate the common and basic features of the cosmological evolution in the thermal string scenario. The problem of moduli stabilization is discussed, in order to motivate the work in the following up two chapters.

In Chapter 7 and Chapter 8 we present the work done in Refs [18, 19], where we focus our attention on the moduli stabilization in the intermediate era by non-perturbative effects. The non-perturbative effects concerned are those mentioned in chapter 4. The D1-brane states in type I string is investigated in chapter 7 while the D2(3)-brane effects in Calabi-Yau compactification of type IIA(B) string are considered in chapter 8. We have already explained the idea in the middle part of Sec.1.3

We will give conclusion and perspectives in the final chapter.

Chapter 2

Perturbative string spectrum

This chapter gives a sketch of important elements in perturbative string theories which we will need. We go quickly through the perturbative quantization of all types of string theories, where we care most about the resulting spectra and their description in terms of partition functions. The latter leads to the computation of one-loop vacuum-vacuum amplitudes, which will play crucial role later in the study of cosmology. Although non-perturbative effects will also be investigated as very important issue, the computation involved therein will still utilize the perturbative technique in this chapter.

2.1 Quantization of free string theories in general

Naively the string theory can be viewed as a generalization of the point particle vision of the microscopic world. It postulates that the fundamental component of matter are extended objects of one spatial dimension, which has internal structure rather than point-like objects. In Fig.2.1 this generalization is shown schematically. In the passage from the point particle action to the bosonic string action, we emphasize the enlargement of local symmetry group due to the internal structure of strings. Handling properly these symmetries in the quantization of string theories leads to nontrivial constraints on the structure of spacetime and on the spectrum that string vibrations generate.

String theories in a curved spacetime background is in general a nonlinear sigma model, of which a full quantum description is not possible except for some special cases. Here for sake of exact quantization, we consider a Minkowskian background, which can be further compactified on some toroidal compact space.

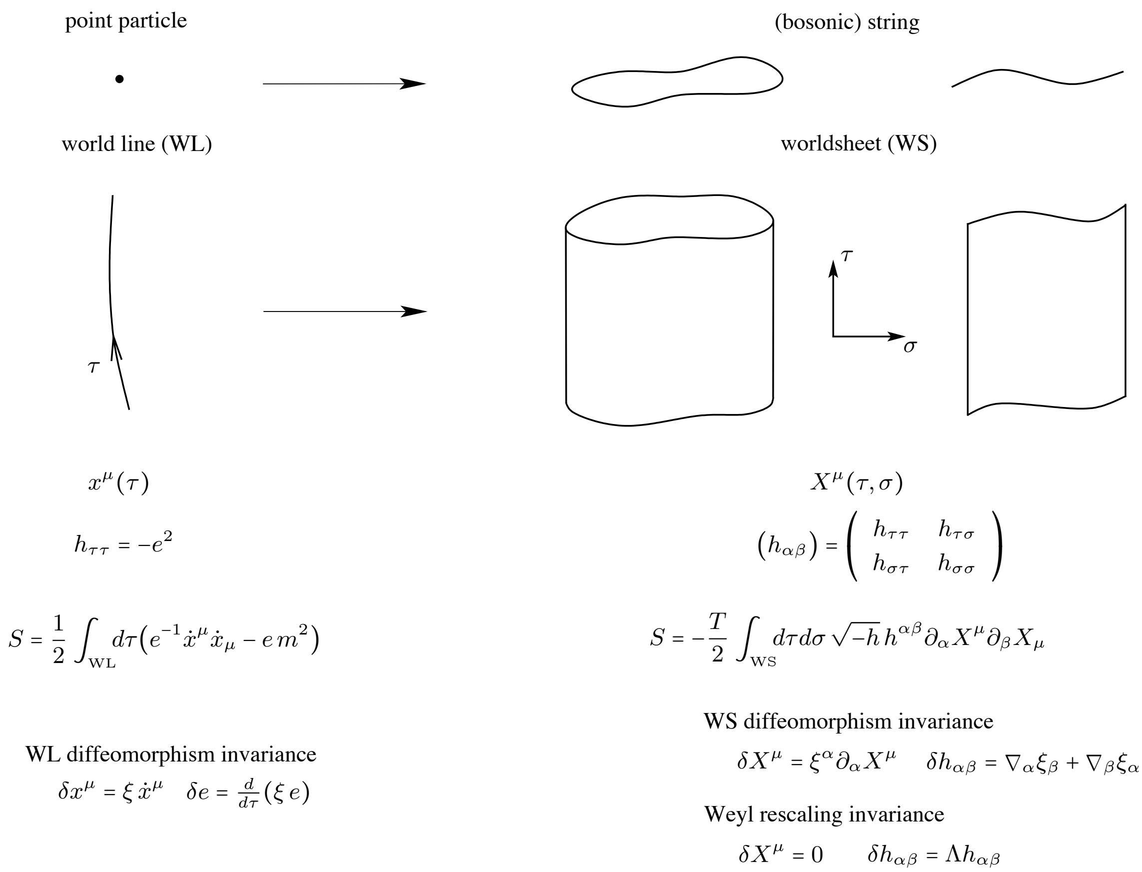


Figure 2.1: From scalar particle to bosonic string. Here μ labels spacetime dimension, α β label the worldsheet dimensions; τ is the propertime of the world line and the worldsheet, and σ the spatial coordinate of the worldsheet; $h_{\tau\tau} = -e^2$ is the world line metric and $h_{\alpha\beta}$ the worldsheet metric; $x^\mu(\tau)$ and $X^\mu(\tau,\sigma)$ are respectively the target space coordinates of the particle and of the string; $T = \frac{1}{2\pi l_s^2}$ is the string tension; ξ is the infinitesimal generator of world line diffeomorpism and ξ^α the worldsheet counterpart; Λ is the infinitesimal generator of Weyl rescaling.

Quantization schemes

To quantize string theories in a Minkowskian target space, one usually go through three different schemes: old covariant quantization, lightcone gauge quantization and BRST quantization. Each has its own domain of competence. In case of non-interacting strings where all these quantization schemes can be applied, they lead to equivalent physical Hilbert spaces related by isomorphisms. All string theory quantization in this chapter will be using lightcone quantization. Here we briefly comment these methods in order to justify our choice, as well as to motivate the CFT approach of string theory quantization.

Old covariant quantization

It is based on canonical quantization scheme, and is intended for giving the physical spectra of free string theories. One starts by imposing conformal (superconformal) gauge condition to bosonic string (superstring) theory, which fixes the worldsheet metric up to a Weyl rescaling. The action thus appears as a free CFT (SCFT), but is supplemented with first-class constraints. The latter are associated to the reparameterization (and local supersymmetry) freedoms of string coordinates. One then quantizes this CFT (SCFT) canonically, and in addition imposes the classical constraints at quantum level to squeeze out gauge redundancies. Weyl symmetry is violated at quantum level due to the presence of a nonzero central charge. However this conformal anomaly is irrelevant, since quantizing the theory against two different worldsheet metrics subject to conformal (superconformal) gauge condition leads to the same Hilbert space. Generically the quantum description suffers from negative-norm states in the Hilbert space, which are called ghosts. By requiring the absence of ghost states, one finds precise constraints on the spacetime dimension and the mass spectrum.

Lightcone quantization

It is also based on canonical quantization and deals only with the free string theories. By imposing the lightcone gauge condition, all reparametrization and local supersymmetry freedoms are completely fixed at classical level. One is thus left with a free 2-dimensional CFT or SCFT containing only transverse string coordinates. Radial quantization yields directly the exact physical Hilbert space, where all physical states span a representation space of the Virasoro algebra. Weyl invariance is not preserved at quantum level since there is a nonzero central charge. However just as in the old covariant quantization, this conformal anomaly is not harmful in itself, while the harm is reincarnated in the Lorentz algebra anomaly in spacetime. Demanding the cancelation of this anomaly, one gets the same constraints on the spacetime dimension and the same mass spectrum as in the former case. In the case where we care only about the free string spectrum, lighcone quantization is the most straightforward and economic way to achieve the goal, as will be the case for our computation of one-loop vacuum amplitudes.

BRST quantization

The BRST quantization invokes path integral method and is designed to rigorously compute string amplitudes to any level in loop expansion. For free string theories, the gauge fixing procedure leads to a 2-dimensional CFT or SCFT containing all the bosonic string coordinates and ghosts, as well as all the fermionic string coordinates with superghosts in case of superstring theory. The action then loses all gauge freedom of worldsheet reparameterization and local supersymmetry but still preserves Weyl rescaling invariance, and in addition it acquires BRST invariance. These two local symmetries are generically not preserved at quantum level. By cancelation of the associated anomalies, the spacetime dimension number and the mass spectrum are constrained in the same way as the former two cases. One can further work out the Hilbert space, where physical states are indicated by its BRST cohomology classes.

Free CFT on complex plane

The canonical quantization of free string theory attributes to the quantization of the underlying free CFT's on a complex plane, whose field contents vary in function of the type of strings and the method of quantization. Let the complex coordinate be z, and the free CFT be invariant under the conformal transforms $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$ with f any holomorphic function. The associated conserved charges are $T(z)$ and $\bar{T}(\bar{z})$, whose operator product expansions (OPE) tell the central charges $(c, \bar{c})$ of the CFT:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \ldots, \tag{2.1}$$

and the same expression for $\bar{T}(\bar{z})$, which implies $\bar{c}$. We will need to know that each holomorphic or anti-holomorphic free scalar contributes one unit of central charge and each free chiral fermion contributes one half. The mode expansions of $T(z)$ and $\bar{T}(\bar{z})$ yield Virasoro operators $\{L_k\}$ and $\{\bar{L}_k\}$: $T(z) = \sum z^{-k-2} L_k$ and $\bar{T}(\bar{z}) = \sum \bar{z}^{-k-2} \bar{L}_k$, which satisfy anomalous Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m-n} + \frac{c}{12}(m^3 - m)\delta_{m+n}, \tag{2.2}$$

and the same thing for $\{\bar{L}_k\}$ with central charge $\bar{c}$. There exists the vacuum state $|0\rangle$ which is annihilated by $L_{k\geq -1}$ and $\bar{L}_{k\geq -1}$. Primary fields $\phi(z, \bar{z})$ are those which transform according to

$$\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{h}{(z-w)^2}\,\phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\bar{w}) + \ldots, \\ \bar{T}(z)\phi(w,\bar{w}) &= \frac{\bar{h}}{(z-w)^2}\,\phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\bar{w}) + \ldots, \end{aligned} \tag{2.3}$$

where h and $\bar{h}$ are real constants, called the conformal weights of ϕ. Ground physical states are generated by primary states as

$$|h, \bar{h}\rangle = \lim_{z,\bar{z}\to 0} :\phi(z,\bar{z}): |0\rangle, \tag{2.4}$$

which is of conformal weight $(h, \bar{h})$, with $L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle$ and $\bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle$. The physical spectrum can be generated by acting the negative modes L_{-k} and $\bar{L}_{-k}$ $(k > 0)$ on the ground state $|h, \bar{h}\rangle$, subject to certain physical constraints, for example the level-matching condition $L_0 = \bar{L}_0$ for closed strings, the GSO conditions, invariance under orbifold projections or orientifold projections, etc. All descendent physical states based on the ground state $|h, \bar{h}\rangle$ span a representation (not necessarily irreducible) space of the Virasoro algebra, of highest weight $(h, \bar{h})$. At this point we can introduce the characteristic function of a Virasoro algebra representation, called conformal character. When holomorphic sector and the anti-holomorphic sector can have independent highest weights and descendent states, as is the case for closed strings, the conformal character based on ground state $(h, \bar{h})$ is

$$\chi_{(h,\bar{h})}(\tau, \bar{\tau}) = \mathrm{Tr}_{(h,\bar{h})} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} = q^{h-\frac{c}{24}} \bar{q}^{\bar{h}-\frac{\bar{c}}{24}} \sum_N D_N q^N \sum_{\bar{N}} \bar{D}_{\bar{N}} \bar{q}^{\bar{N}}, \tag{2.5}$$

where $q = e^{2\pi i \tau}$, and τ is a complex parameter, which is defined on the upper complex plane to guarantee the convergence of the series; the trace $\mathrm{Tr}_{(h,\bar{h})}$ runs through all the descendent states based on the ground state $|h, \bar{h}\rangle$, and D_N $(\bar{D}_{\bar{N}})$ is the degeneracy of the N-th ($\bar{N}$-th) oscillator level in the holomorphic (anti-holomorphic) sector. For the state $L_{-m_1} \ldots L_{-m_k} \bar{L}_{-n_1} \ldots \bar{L}_{-n_l} |h, \bar{h}\rangle$, its oscillator levels are $N = m_1 + \cdots + m_k$ and $\bar{N} = n_1 + \cdots + n_l$. Here we include the central charge shift $-c/24$ and $-\bar{c}/24$, in order to switch from the complex plane back to the initial worldsheet of free string theories. Often we need to consider the holomorphic and anti-holomorphic sectors separately, and we can define their conformal characters respectively as

$$\chi_h(\tau) = q^{h-\frac{c}{24}} \sum_N D_N q^N ; \qquad \bar{\chi}_{\bar{h}}(\bar{\tau}) = \bar{q}^{\bar{h}-\frac{\bar{c}}{24}} \sum_{\bar{N}} \bar{D}_{\bar{N}} \bar{q}^{\bar{N}}. \tag{2.6}$$

When modeling free open string theories, the CFT is defined on the upper half complex plane and the holomorphic sector is identified with the anti-holomorphic sector: $T(z) \equiv \bar{T}(\bar{z}) \Leftrightarrow L_m \equiv \bar{L}_m$. In such case a generic state is $L_{-n_1} \ldots L_{-n_k} |h\rangle$ $(n_1, \ldots, n_k > 0)$ and the corresponding conformal character is

$$\chi_h(\tau, \bar{\tau}) = \mathrm{Tr}_h q^{L_0 - \frac{c}{24}} = q^{h-\frac{c}{24}} \sum_N D_N q^N, \tag{2.7}$$

where $q = e^{-2\pi \mathrm{Im}\tau}$. The trace goes through all descendent oscillator states based on $|h\rangle$. Therefore Eq.(2.5) is the characteristic function of spectrum on the cylinder and Eq.(2.7) is the characteristic function of spectrum on the strip. They are closely related to string one-loop vacuum amplitudes, as is to be explained next.

One-loop vacuum amplitude

The computation of one-loop vacuum amplitudes in string theories can be summarized as

$$Z_1 = \int_{D_{\rm F}} d\mu(\tau,\bar\tau)\,\mathscr{A}_1(\tau,\bar\tau), \tag{2.8}$$

where τ denotes collectively the parameters characterizing the geometric configuration of the worldsheet concerned, referred to as the Teichmüler parameter, which in our case can be identified with the τ appearing in the conformal character Eqs (2.5) and (2.7); $\mathscr{A}_1(\tau,\bar\tau)$ is the one-loop vacuum amplitude computed against a specific geometric configuration τ; and $D_{\rm F}$ is the minimum parametric space of τ containing all distinct configurations. Finally all the specific amplitudes $\mathscr{A}_1(\tau,\bar\tau)$ are integrated up with measure $d\mu(\tau,\bar\tau)$, to give the total amplitude Z_1. The amplitude $\mathscr{A}_1(\tau,\bar\tau)$ is model-dependent, which requires knowledge of the physical modes circulating in the loop. Since we are dealing with non-interacting strings, all the three quantization schemes can work out $\mathscr{A}_1(\tau,\bar\tau)$. The other two elements $d\mu(\tau,\bar\tau)$ and $D_{\rm F}$ do not care about the physical modes on the worldsheet, but care only about the worldsheet topology. Therefore only BRST quantization is qualified for finding them out, and hence qualified for computing Eq.(2.8) rigorously from A to Z. However, we will encounter only four types of topology at one-loop level, namely the torus, the Klein bottle, the annulus and the Möbios strip, whose $d\mu(\tau,\bar\tau)$ and $D_{\rm F}$ are already known. Thus we will simply take and use these results, and focus our attention on the computation of $\mathscr{A}_1(\tau,\bar\tau)$. The latter can be most conveniently achieved with lightcone gauge quantization. Thus $\mathscr{A}_1(\tau,\bar\tau)$ is just the conformal character of the lightcone Hilbert space, summed up over all possible highest weighs (integrate over continuous highest weighs). For closed and open strings, we have respectively,

$$\begin{aligned} \text{closed}: \quad & \mathscr{A}_1(\tau,\bar\tau) = \sum\!\!\!\!\!\!\int_{h,\bar h} \text{sign}(h,\bar h) \times \chi_{(h,\bar h)}(\tau,\bar\tau)\,, \\ \text{open}: \quad & \mathscr{A}_1(\tau,\bar\tau) = \sum\!\!\!\!\!\!\int_{h} \text{sign}(h) \times \chi_{h}(\tau,\bar\tau)\,. \end{aligned} \tag{2.9}$$

The subtlety in the summation is that appropriate sign should be attributed to different characters, here denoted by sign(...), in order to implement physical conditions, for example GSO projection, modular invariance. It can be seen from Eqs (2.5) and (2.7) that $\mathscr{A}_1(\tau,\bar\tau)$ can be expanded as a sum of powers of q and $\bar q$. In case of closed string amplitude, the expansion containing both $q = e^{2\pi i\tau}$ and $\bar q = e^{-2\pi i\bar\tau}$, physical spectrum can be read off from the level-matched part (q and $\bar q$ have the same power) of this expansion, with the powers of $q\bar q$ giving the squared masses, and the associated coefficient the degeneracy of this mass level. It should be mentioned that the level-matching condition is not derived from the underlying CFT, but is inherited from the worldsheet diffeomorphism invariance of the full string theory action. However the modes which do not satisfy level-matching condition contribute to the closed string one-loop amplitude even though they are not in the free string spectrum. On the other hand for open strings, the expansion of $\mathscr{A}_1(\tau,\bar{\tau})$ is uniquely in terms of $q = e^{-2\pi \mathrm{Im}\tau}$, and the physical spectrum is read off from this expansion in the way that the power of q imply the masses squared, the coefficient is the degeneracy of this mass level. Finally we recall without proving that in the frame work of lightcone gauge the integral measures in Eq.(2.8) are

$$\frac{V_D}{2(2\pi)^D}\int_{\mathcal{F}}\frac{d\tau_1 d\tau_2}{\tau_2^2}, \quad \text{with} \quad \tau = \tau_1 + i\,\tau_2 \quad \text{(torus)}, \tag{2.10}$$

$$\frac{V_D}{2(2\pi)^D}\int_0^\infty \frac{d\tau_2}{\tau_2^2} \quad \text{with} \quad \tau = \begin{cases} 2i\tau_2 & \text{(Klein bottle)},\\ \frac{1}{2}i\tau_2 & \text{(annulus)},\\ \frac{1}{2}+\frac{1}{2}i\tau_2 & \text{(Möbius strip)}. \end{cases} \tag{2.11}$$

In the torus case, the integration domain $\mathcal{F}$ is defined by $|\tau| > 1$ and $-\frac{1}{2} < \tau_1 < \frac{1}{2}$, which is the fundamental domain of $SL(2,\mathbb{Z})$. Indeed the integral measure $\tau_2^{-2} d\tau_1 d\tau_2$ is invariant under the transform $\tau \to \frac{a\tau+b}{c\tau+d}$, with $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$.

2.2 Bosonic string

The bosonic string lives in 26 dimensional flat background spacetime. The classical Polyakov action, which describes 26 scalar fields $\{X^\mu(\tau,\sigma)\}$ ($\mu = 0,\dots,25$) on the worldsheet, and which displays all symmetries, is already shown in Fig.2.1. The notation of worldsheet time τ is in clash with the Teichmüler parameter in Eq.(2.8), but since we will never use the two together, there will be no risk of confusion. Imposing lightcone gauge condition which eliminates X^0 and X^1, performing worldsheet Wick rotation $\tau = -i\tau_{\mathrm{E}}$ and defining the complex coordinate $z = e^{\tau_{\mathrm{E}}-i\sigma}$, we reduce the initial gauge system to a 2D CFT on the z-plane with 24 transverse free scalar fields $\{X^{\tilde{\imath}}(z,\bar{z})\} = \{X^i, X^I\}$. Here we suppose that the theory is compactified on T^d, and let $i = 2,\dots,D-1$ label non compact spacetime of dimension $D = 26-d$, and $I = D,\dots,25$ label compact ones. The action is

$$S^{\mathrm{l.c.}} = \frac{T}{2}\int d^2z\,\partial X^{\tilde{\imath}}\bar{\partial}X^{\tilde{\imath}}, \tag{2.12}$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$, and by d^2z we mean $d\mathrm{Re}z\,d\mathrm{Im}z$. The string tension T is related to the string length l_s and to the Regge slope α' by $T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2}$. The energy-momentum tensor is

$$T^X(z) = -\frac{1}{l_s^2}\,\partial X^{\tilde{\imath}}\partial X^{\tilde{\imath}}, \quad \bar{T}^X(\bar{z}) = -\frac{1}{l_s^2}\,\bar{\partial}X^{\tilde{\imath}}\bar{\partial}X^{\tilde{\imath}} \tag{2.13}$$

with central charge $(c^X, \bar{c}^X) = (24, 24)$, and we let the associated Virasoro operators be $\{L^X_k\}$ and $\{\bar{L}^X_k\}$. The quantization of this system follows the standard scheme for free CFT quantization. The following contents adopts the language in [27] and will be very concise on mathematical aspects of CFT.

Closed bosonic string

The string coordinates satisfy periodic boundary conditions $X^{\tilde{i}}(e^{2\pi i}z) = X^{\tilde{i}}(z)$, whose mode expansion is

$$
\begin{aligned}
&X^{\tilde{i}}(z,\bar{z}) = X^{\tilde{i}}_L(z) + X^{\tilde{i}}_R(\bar{z}), \quad \text{with} \\
&X^i_L = \frac{1}{2}x^i - i\,\frac{l_s^2}{2}\,k^i \ln z + i\frac{l_s}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\frac{\alpha^i_n}{z^n}, \quad X^i_R = \frac{1}{2}x^i - i\,\frac{l_s^2}{2}\,k^i \ln z + i\frac{l_s}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\frac{\bar{\alpha}^i_n}{\bar{z}^n}; \\
&X^I_L = \frac{1}{2}x^I_L - i\,\frac{l_s^2}{2}\,p^I_L \ln z + i\frac{l_s}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\frac{\alpha^I_n}{z^n}, \quad X^I_R = \frac{1}{2}x^I_R - i\,\frac{l_s^2}{2}\,p^I_R \ln \bar{z} + i\frac{l_s}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\frac{\bar{\alpha}^I_n}{\bar{z}^n}.
\end{aligned}
\tag{2.14}
$$

The zero modes are of interest since they contain physical information that the underlying CFT does not imply. Note that for compact directions the holomorphic and anti-holomorphic components have independent center-of-mass positions and momenta, $\{x^I_L, x^I_R\}$ and $\{p^I_L, p^I_R\}$. It is because the closed strings can wrap around compact directions. All the center-of-mass position and momentum components have non-trivial commutation relations:

$$
[x^i, k^j] = i\,\delta^{ij}, \quad [x^I_{L,R}, p^J_{L,R}] = i\,\delta^{IJ}, \quad 0 \text{ for other commutators.} \tag{2.15}
$$

The CFT vacuum state $|0\rangle$ is annihilated by all positive α-oscillators, and it can be shown that it is SL_2 invariant, i.e. annihilated by $L_{0,\pm 1}$ and $\bar{L}_{0,\pm 1}$. A generic physical ground state is obtained by acting on the vacuum the primary field

$$
\mathcal{O}^{\text{cls}}_{k,p}(z,\bar{z}) =: e^{i\,k^i X^i(z,\bar{z})}\, e^{i\,p^I_L X^I_L(z) + i\,p^I_R X^I_R(\bar{z})} :, \tag{2.16}
$$

where the superscript "cls" stands for "closed", and k^i and $p^I_{L,R}$ here are not operators but take concrete values. The OPEs of $V_{k,p}(z,\bar{z})$ with $T^X(z)$ and $\bar{T}^X(\bar{z})$ tell that the conformal weight is

$$
h^X = \frac{l_s^2}{4}\left(k^2 + p_L^2\right), \quad \bar{h}^X = \frac{l_s^2}{4}\left(k^2 + p_R^2\right), \tag{2.17}
$$

so is the conformal weight of the corresponding ground states. The center-of-mass momentum in non compact directions k^i can be of any value, while k^- adjusts itself to ensure $k^+k^- - k^i k^i$ onshell. However the values of $p^I_{L,R}$ are constrained by the compact space, which contain Kaluza-Klein (KK) and winding numbers in the compact directions, as well as the moduli arising from compactification. A generic physical state $|X_{\rm cls}\rangle$ can be the ground state or oscillator excitation based on it:

$$|X_{\rm cls}\rangle = ({\rm osc.})\, \mathcal{O}^{\rm cls}_{k,p}(0,0)|0\rangle = \big|({\rm osc.})\,; p_L, p_R; k\,\big\rangle. \tag{2.18}$$

where "osc." denotes collectively negative-level oscillators. These states, with specific k and $p_{L,R}$, fit into a representation of the complex Virasoro algebra constituted by $\{L^X_k, \bar{L}^X_k\}$, of highest weight $(h^X, \bar{h}^X)$ as shown in (2.17). Especially acting zero modes of the Virasoro operators on them gives

$$L^X_0|X_{\rm cls}\rangle = \Big[\frac{l_s^2}{4}\left(k^2+p_L^2\right)+N\Big]|X_{\rm cls}\rangle, \quad \bar{L}^X_0|X_{\rm cls}\rangle = \Big[\frac{l_s^2}{4}\left(k^2+p_R^2\right)+\bar{N}\Big]|X_{\rm cls}\rangle, \tag{2.19}$$

with N and $\bar{N}$ non-negative integers denoting left and right oscillator levels. We recall the result from CFT that the tower of oscillator excitations of a single holomorphic worldsheet boson is characterized by the partition function $q^{\frac{1}{24}}/\eta(\tau)$, where the expansion coefficient of the N-th power of $q = e^{2\pi i\tau}$ is the degeneracy of the oscillator level N. In the same way we have the anti-holomorphic partition function $\bar{q}^{\frac{1}{24}}/\bar{\eta}(\bar{\tau})$. Thus using Eq.(2.19), one finds the conformal character of the spectrum Eq.(2.18) to be

$$\chi^X_{k,p}(\tau,\bar{\tau}) = {\rm Tr}\, q^{L_0-\frac{c^X}{24}}\,\bar{q}^{\bar{L}_0-\frac{\bar{c}^X}{24}} = q^{\frac{l_s^2}{4}(k^2+p_L^2)}\bar{q}^{\frac{l_s^2}{4}(k^2+p_R^2)}\,\eta(\tau)^{-24}\bar{\eta}(\bar{\tau})^{-24}, \tag{2.20}$$

where the trace runs through all the oscillator excitations. Referring to the first line in Eq.(2.9) the total one-loop amplitude in vacuum is therefore

$$Z = \frac{V_D}{2(2\pi)^D}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{p_L,p_R}\int d\vec{k}\;\chi^X_{k,p}(\tau,\bar{\tau}) = \frac{V_D}{2(2\pi)^D}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}}\sum_{p_L,p_R}\frac{q^{\frac{l_s^2}{4}p_L^2}\bar{q}^{\frac{l_s^2}{4}p_R^2}}{\eta(\tau)^{24}\,\bar{\eta}(\bar{\tau})^{24}}\,. \tag{2.21}$$

The expansion of the integrand yields the lowest order $(q\bar{q})^{-1}$ with negative power, showing that the ground state is a tachyon. Also one can show that all states on the same mass level form a tensorial representation of the little group. Therefore the spectrum Eq.(2.18) contains no spacetime fermions. These drawbacks decide that the bosonic string cannot lead to sensible phenomenology.

Open bosonic string

The worldsheet scalar fields satisfy the Neumann boundary conditions $\partial_\sigma X^i\big|_{\sigma=0,\pi} = 0$, and we have the mode expansions

$$X^i(z,\bar{z}) = \frac{1}{2}x^i - i\,l_s^2\,k^i\ln z\bar{z} + ({\rm osc.}), \quad X^I = \frac{1}{2}x^I - i\,l_s^2\,p^I\ln z\bar{z} + ({\rm osc.}) \tag{2.22}$$

where the nonzero oscillator parts are the same to Eq.(2.14) but with all left-moving and right-moving oscillators identified: $\alpha_*^{\tilde{\imath}} = \bar{\alpha}_*^{\tilde{\imath}}$. Especially the center-of-mass momenta and positions in compact directions are no longer split into left-moving and right-moving parts, since open strings cannot wrap around compact directions, and hence in the internal momenta p^I there is no winding number. Also the center-of-mass momenta are defined differently from the case of closed string, to maintain the correct commutation relation with center-of-mass positions. The CFT vacuum is still denoted by $|0\rangle$, and a generic physical ground state is generated by the primary field

$$\mathcal{O}^{\text{op}}_{k,p}(z,\bar{z}) = :e^{i k^i X^i(z,\bar{z})}\, e^{i p^I X^I(z,\bar{z})}:, \tag{2.23}$$

where k^i can take any value and p^I are constrained by the geometry of compact directions. It has conformal weight

$$h^X = l_s^2\,(k^2 + p^2). \tag{2.24}$$

A generic physical state $|X_{\text{op}}\rangle$ is either a ground state generated by Eq.(2.23) or is a descendent state with oscillator excitations, which is

$$|X_{\text{op}}\rangle = (\text{osc.})\,\mathcal{O}^{\text{op}}_{k,p}(0,0)\,|0\rangle \otimes |ij\rangle = \big|(\text{osc.}); p; k; ij\big\rangle, \tag{2.25}$$

satisfying

$$L_0^X|X_{\text{op}}\rangle = \big[l_s^2\,(k^2+p^2) + N\big]|X_{\text{op}}\rangle, \tag{2.26}$$

where N denotes collectively oscillator excitations; $|ij\rangle$ is the sector carrying Chan-Paton factors i and j, which is a non-dynamical sector introduced in order to implement non-Abelian gauge symmetry in spacetime. These states supply a representation of a Virasoro algebra of highest weight h^X as in Eq.(2.24). The one-loop amplitude is

$$Z = \frac{V_D}{2(2\pi)^D}\int_0^\infty \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}}\sum_p \frac{q^{\frac{l_s^2}{2}p^2}}{\eta(2i\tau_2)^{24}}, \quad \text{where} \quad q = e^{-4\pi\tau_2}. \tag{2.27}$$

This integral has UV divergence at the limit $\tau_2 \to 0$. Although the pathology can be cured by including closed string and implementing the orientifold projection, the theory still suffers from the problems of tachyonic ground state and lack of spacetime fermion. Therefore bosonic strings cannot be a phenomenologically viable theory and we move on to superstring theories.

2.3 Closed $\mathcal{N} = (1,1)_2$ superstring and type II strings

The $\mathcal{N} = (1,1)_2$ has 10 dimensional target space. Its classical action is the supersymmetrized version of the Polyakov action for bosonic strings

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}\left\{h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + 2i\,\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu + (\text{worldsheet gravitino})\right\}, \tag{2.28}$$

where $\{\rho^\alpha\}$ are Dirac matrices in 2 dimensions. It is in effect a 2-dimensional supergravity with a non-dynamical supergravity multiplet, 10 scalar fields $\{X^\mu(\tau,\sigma)\}$ and 10 2-component Majorana spinors $\{\psi^\mu(\tau,\sigma)\}$. By lightcone gauge fixation, followed by a Wick rotation $\tau = -i\tau_{\mathrm{E}}$ and a coordinate change $z = e^{\tau_{\mathrm{E}}-i\sigma}$, the original theory is brought down to an $\mathcal{N} = (1,1)_2$ SCFT on the z-plane. It contains 8 free scalar fields $\{X^{\tilde{\imath}}\}$ and their superpartners which are free Majorana-Weyl spinors $\{\psi^{\tilde{\imath}},\bar{\psi}^{\tilde{\imath}}\}$ ($i = 2,\dots,9$), with ψ and $\bar{\psi}$ of opposite chirality. It is a free SCFT of central charge $(c^{X,\psi},\bar{c}^{X,\bar{\psi}}) = (12,12)$ where each holomorphic boson contribute one unit and fermion half unit. The action is:

$$S^{\text{l.c.}} = \frac{T}{2}\int d^2z\left[\partial X^{\tilde{\imath}}\bar{\partial}X^{\tilde{\imath}} + i\left(\psi^{\tilde{\imath}}\bar{\partial}\psi^{\tilde{\imath}} + \bar{\psi}^{\tilde{\imath}}\partial\bar{\psi}^{\tilde{\imath}}\right)\right], \tag{2.29}$$

from which we derive the conserved charges of conformal transforms

$$T^{X,\psi}(z) = -\frac{1}{l_s^2}\,\partial X^{\tilde{\imath}}\partial X^{\tilde{\imath}} + \frac{1}{l_s^2}\psi^{\tilde{\imath}}\,\partial\psi^{\tilde{\imath}}, \quad \bar{T}^{X,\bar{\psi}}(\bar{z}) = -\frac{1}{l_s^2}\,\bar{\partial}X^{\tilde{\imath}}\bar{\partial}X^{\tilde{\imath}} + \frac{1}{l_s^2}\bar{\psi}^{\tilde{\imath}}\,\bar{\partial}\bar{\psi}^{\tilde{\imath}}. \tag{2.30}$$

Here as for the bosonic string, we suppose the theory be compactified on a torus T^d, where $d = 10 - D$ and D the spacetime dimension. The convention for the indices are $i = 2,\dots,D-1$ and $I = D,\dots,9$. The quantization is straightforward: the bosonic part and the fermionic part can be quantized as independent CFT's. The bosonic part has already been described in the last section, and the fermionic part is summarized as follows.

Spectrum of worldsheet fermions

Unlike the bosonic sector, the fermionic sector gives rise to Virasoro algebra representations of fixed highest weights. Recall from Ref. [27] that each free chiral worldsheet fermion can generate the representation $[0] + 2[\frac{1}{16}] + [\frac{1}{2}]$, where the numbers in the brackets are the highest weights of irreducible components. The representation $[0] + [\frac{1}{2}]$ is generated when the chiral fermion takes anti-periodic boundary condition, referred to as Neveu-Schwarz (NS) boundary condition. This leads to integer mode expansion of worldsheet fermions $\psi^{\tilde{\imath}}(z) = \sum_r z^{-r-\frac{1}{2}}\psi^{\tilde{\imath}}_r$ ($r \in \mathbb{Z} - \frac{1}{2}$), and the conformal character of the resulting spectrum is[1]

$$\chi^{\psi}\big([0]+[\tfrac{1}{2}]\big)=\sqrt{\frac{\theta_3}{\eta}}\,. \tag{2.31}$$

The representation $[\frac{1}{16}]$ is generated when the chiral fermion takes periodic boundary condition, referred to as Ramond (R) boundary condition. This results in half-integer mode expansion of worldsheet fermions $\psi^{\tilde{\imath}}(z)=\sum_r z^{-r-\frac{1}{2}}\psi^{\tilde{\imath}}_r$ $(r\in\mathbb{Z})$, where the the zero modes $\psi^{\tilde{\imath}}_0$ are crucial to the emergence of spacetime fermions. Each $[\frac{1}{16}]$ representation in the holomorphic sector is characterized by conformal character

$$\chi^{\psi}\big([\tfrac{1}{16}]\big)=\sqrt{\frac{\theta_2}{2\eta}}\,. \tag{2.32}$$

For sake of global existence of worldsheet supercurrent, fermions of same chirality should take the same boundary condition. The left-moving and the right-moving fermions can be quantized independently. The spectrum of the 8 left-moving fermions is summarized as follows, where according to the choice boundary condition, the Hilbert space is split into the NS sector and the R sector.

NS sector

The NS sector is the $\big([0]+[\frac{1}{2}]\big)^8$ representation of the Virasoro algebra, which contains the CFT vacuum and all the oscillator excitations with half-integer oscillators. The spectrum has conformal character

$$\chi^{\psi}\big([0]+[\tfrac{1}{2}]\big)^8=\frac{\theta_3^4}{\eta^4}\,. \tag{2.33}$$

The spectrum (2.33) does not lead to sensible physical specturm, because it contains a tachyonic ground state. To fix this problem, we require a truncation of the spectrum by the GSO projection[2]. In CFT language, one requires in the NS sector that physical states be those of integer highest weights among the irreducible components in $\big([0]+[\frac{1}{2}]\big)^8$. In the oscillator language, it amounts to requiring an odd number of ψ-oscillator excitations. The conformal character associated to this truncated spectrum is therefore

$$\chi^{\mathrm{NS}}_{\mathrm{GSO}}=\frac{1}{2}\left(\frac{\theta_3^4}{\eta^4}-\frac{\theta_4^4}{\eta^4}\right)=V_8. \tag{2.34}$$

[1]For the elliptic η- and θ-functions, we adopt the conventions and notations in the appendix C of [30] for elliptic θ-functions.

[2]Attributed to Gliozzi, Scherk and Olive.

Here the second term in the parenthesis θ_4^4/η^4 is computed as $\chi^{\psi}\big([0]+[\frac{1}{2}]\big)^8$, but all contribution from representations of half-integer highest weight has the sign reversed. Thus the subtraction of the two projects out the representations of integer highest weight representations.

R sector

The R sector of the Hilbert space is a $\big(2[\frac{1}{16}]\big)^8$ representation of Virasoro algebra. It contains a 2^8-fold ground state of conformal weight $\frac{8}{16}$, which forms an $SO(8)$ Dirac spinor in spacetime. The zero modes in the oscillator expansion $\psi_0^{\tilde{\imath}}$ act on this ground state as Dirac matrices since they satisfy the Clifford algebra $\{\psi_0^{\tilde{\imath}}, \psi_0^{\tilde{\jmath}}\} = \delta^{\tilde{\imath}\tilde{\jmath}}$. The raw R sector spectrum contains four times the amount of NS sector modes. To achieve spacetime supersymmetry, we need to truncate the spectrum. First we remove half of the modes by imposing the Majorana condition, making the ground state a real spinor. We can thus express explicitly the ground state using bosonization formalism of worldsheet fermions, where this ground state is created out of the CFT vacuum by spin fields[3]. Second we introduce the GSO condition for the R sector by requiring that physical R sector states should be of a definite chirality, and furthermore, two adjacent oscillator excitations should have opposite chirality. The associated conformal character is[4]

$$\chi^{\mathrm{R}}_{\mathrm{GSO}} = \frac{1}{2}\left(\frac{\theta_2^4}{\eta^4} \pm \frac{\theta_1^4}{\eta^4}\right) = \begin{cases} S_8 & + \,, \\ C_8 & - \,. \end{cases} \tag{2.35}$$

Here the first term in the parenthesis θ_2^4/η^4 is nothing but $\chi^{\psi}\big(16[\frac{1}{16}]^8\big)$, the conformal character arising from the spectrum based on a Majorana ground state. The second term is obtained, based on the first term, by reversing the sign of the contribution from one chirality. The subtlety is that when going from one oscillator excitation to an adjacent one, the sign should be reversed for opposite chirality. The sign ambiguity in Eq.(2.35) accounts for the fact that one can project out any of the two chiralities.

Above is the quantization of left-moving fermions. The total conformal character is

$$\begin{aligned} \chi^{\psi}(\tau) =& \chi^{\mathrm{NS}}_{\mathrm{GSO}}(\tau) - \chi^{\mathrm{R}}_{\mathrm{GSO}}(\tau) = \frac{\theta_3^4 - \theta_4^4 - \theta_2^4 \pm \theta_1^4}{2\eta^4} \\ =& \sum_{a,b=0}^{1} (-1)^{a+b+\mu\, ab} \frac{\theta[^a_b]^4}{2\eta^4} = \begin{cases} V_8 - C_8 & \mu = 0, \\ V_8 - S_8 & \mu = 1. \end{cases} \end{aligned} \tag{2.36}$$

[3]Refer to, for instance, [28] for explicit construction of spin fields through bosonization of worldsheet fermions. A concise description can be found in the beginning of chapter 13 of [29]. With bosonization it is then straightforward to compute the conformal weight of the ground state using the equivalent free bosonic CFT.

[4]The $SO(2n)$-characters $\{O_n, V_n, S_n, C_n\}$ describe respectively the spectra whose ground states transform in the scalar, vector and chiral spinor representations of $SO(2n)$. The notations are introduced in [31].

Here the R sector contribution takes minus sign because it represents fermionic contribution. In the second line, the index a takes value 0 and 1 for NS and R sector, and for a fixed a the sum over b effectuates the GSO projection. In the path-integral approach, a and b indicate the spin structures of the worldsheet fermions. The right-moving worldsheet fermions are quantized in the same way, and what we should obtain is an exact copy of the above results. It can be considered as obtained by adding "$\bar{\ }$" wherever needed in Eqs (2.32)–(2.36).

Type II strings

Now we are at the point of building up the type II strings by assembling the results for worldsheet bosons and fermions of the SCFT (2.29). Tensoring up the bosonic and fermonic Hilbert spaces, we have the whole spectrum divided into four sectors according to the boundary conditions that the left-moving and the right-moving fermions take. These are NS-NS, R-R, NS-R and R-NS sectors, where the first two sectors yield spacetime bosons while the NS-R and R-NS sectors yield spacetime fermions. Now a generic type II state should take one of the following forms:

$$|X_{\text{cls}}\rangle \otimes \left\{ \begin{array}{c} |\text{NS}\rangle_\psi \\ |\text{R}\rangle_\psi \end{array} \right\} \otimes \left\{ \begin{array}{c} |\overline{\text{NS}}\rangle_{\bar{\psi}} \\ |\overline{\text{R}}\rangle_{\bar{\psi}} \end{array} \right\}. \tag{2.37}$$

Here in the braces the upper components are NS sector states and the lower components R sector states, both satisfying the GSO conditions. Finally, physical states should be those among Eq.(2.37) satisfying the level matching condition. The sign ambiguity in the R sector GSO projection leads to two distinct type II string theories: the type IIA (IIB) string arises, when the left-moving and the right-moving R sectors have the opposite (same) chirality for their ground states. To obtain the conformal character of type II spectrum, we gather the results for the bosonic sector Eq.(2.20) and for the sermonic sector Eq.(2.36), and take the product to obtain

$$\begin{aligned}\chi_{\text{II}}(\tau,\bar{\tau}) &= \chi^X_{k,p}(\tau,\bar{\tau})\big[\chi^{\text{NS}}_{\text{GSO}}(\tau) + \chi^{\text{R}}_{\text{GSO}}(\tau)\big]\big[\bar{\chi}^{\overline{\text{NS}}}_{\text{GSO}}(\bar{\tau}) + \bar{\chi}^{\overline{\text{R}}}_{\text{GSO}}(\bar{\tau})\big] \\ &= \left[\frac{q^{\frac{l_s^2}{4}(k^2+p_L^2)}\bar{q}^{\frac{l_s^2}{4}(k^2+p_R^2)}}{\eta^8\bar{\eta}^8}\right]_X \left[\sum_{a,b}(-1)^{a+b+ab}\frac{\theta\big[{a \atop b}\big]^4}{2\eta^4}\right]_\psi \left[\sum_{\bar{a},\bar{b}}(-1)^{\bar{a}+\bar{b}+\mu\,\bar{a}\bar{b}}\frac{\bar{\theta}\big[{\bar{a} \atop \bar{b}}\big]^4}{2\bar{\eta}^4}\right]_{\bar{\psi}},\end{aligned} \tag{2.38}$$

where $\mu = 0$ or 1 correspond to the type IIA or IIB string, and the subscripts X, ψ and $\bar{\psi}$ are for indicating the sector that the conformal characters in the brackets come from. Note that different from the bosonic string we have only eight worldsheet bosons and they give $\eta^{-8}\bar{\eta}^{-8}$. Due to the fact that $a, \bar{a}$ take the value 0 or 1 for NS or R sector, $a + \bar{a}$ is spacetime fermion number with $a + \bar{a} = 0 \bmod 2$ for spacetime bosons and $a + \bar{a} = 1 \bmod 2$ for spacetime fermions. Referring to Eqs

(2.8) and (2.9), one can write down the total vacuum one-loop amplitude

$$
\begin{aligned}
Z_{\mathrm{II}} &= \frac{V_D}{2(2\pi)^D}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}}\sum_{p_L,p_R}\frac{q^{\frac{l_s^2}{4}p_L^2}\bar{q}^{\frac{l_s^2}{4}p_R^2}}{\eta^8\bar{\eta}^8}\sum_{a,b}(-1)^{a+b+ab}\frac{\theta[{}^a_b]^4}{2\eta^4}\sum_{\bar{a},\bar{b}}(-1)^{\bar{a}+\bar{b}+\mu\bar{a}\bar{b}}\frac{\bar{\theta}[{}^{\bar{a}}_{\bar{b}}]^4}{2\bar{\eta}^4} \\
&= \begin{cases} \dfrac{V_D}{2(2\pi)^D}\displaystyle\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}}\frac{\Gamma_{(d,d)}}{\eta^8\bar{\eta}^8}\big(V_8-S_8\big)\big(\bar{V}_8-\bar{C}_8\big), & \text{IIA } (\mu=0) \\ \dfrac{V_D}{2(2\pi)^D}\displaystyle\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}}\frac{\Gamma_{(d,d)}}{\eta^8\bar{\eta}^8}\big(V_8-S_8\big)\big(\bar{V}_8-\bar{S}_8\big). & \text{IIB } (\mu=1) \end{cases}
\end{aligned}
\tag{2.39}
$$

Here we use the shorthand notation $\Gamma_{(d,d)}$ for the Narain lattice sum $\sum q^{\frac{l_s^2}{4}p_L^2}\bar{q}^{\frac{l_s^2}{4}p_R^2}$ corresponding to the T^d compactification, where $d = 10 - D$, and we will always use this notation. The exact form of the internal radii, when all moduli are set free, is

$$
p^I_{L,R} = \Big(m_\alpha - l_s^{-2}n^\beta B_{\alpha\beta}\Big)\,e^{*\alpha I} \mp l_s^{-2}\,n^\alpha e^I_\alpha \tag{2.40}
$$

Here we suppose that the basis of the d-dimensional internal lattice is $\{e^I_\alpha\}$ with $\alpha = D,\ldots,9$ labeling distinct vectors, and $\{e^{*\alpha}_I\}$ the dual lattice basis. The internal metric is obtained by $g_{\alpha\beta} := e^I_\alpha e^I_\beta$. The amplitude (2.39) is modular invariant, thanks to the GSO projection. Expansion of the integrand shows that the ground state is massless so that the spectrum is tachyon free. Numerically the above one-loop amplitude vanishes since we have $V_8 = C_8 = S_8$. This is in fact the consequence of spacetime supersymmetry, where the contribution from spacetime bosons cancels exactly that from spacetime fermions. At ground level, the spectrum contains a graviton in the NS-NS sector and two gravitini in the NS-R and the R-NS sector respectively. In case of maximal number of supersymmetry, it has 32 supercharges. The low energy effective theory in 10 dimensions is the $\mathcal{N}_{10} = 2$ Abelian supergravity. However for phenomenological application, one should implement non Abelian gauge symmetry, and one way to achieve this goal is to construct the type I string or the heterotic string.

Unoriented closed string

As preliminary knowledge for type I string, we anticipate here the notion of unoriented string. The worldsheet parity operator Ω, which reverses string orientation, is defined on the classical level such that $\Omega : \sigma \to 2\pi - \sigma$ and the worldsheet time τ left invariant. At quantum level, it sends a left-moving oscillator to its right-moving counterpart: $p^I_L \xleftrightarrow{\Omega} p^I_R$, $\alpha^{\tilde{\imath}}_n \xleftrightarrow{\Omega} \bar{\alpha}^{\tilde{\imath}}_n$, $\psi^{\tilde{\imath}}_r \xrightarrow{\Omega} \bar{\psi}^{\tilde{\imath}}_r \xrightarrow{\Omega} \pm\psi^{\tilde{\imath}}_r$ (+/− for NS/R sector) etc, and it acts trivially on the vacuum state. Thus its action on a generic state reverses the left and right excitations:

$$
\Omega\,\big|\mathrm{Left},\mathrm{Right};k\,\big\rangle = \pm\big|\overline{\mathrm{Right}},\overline{\mathrm{Left}};k\,\big\rangle, \tag{2.41}
$$

where $\left|\text{Left}, \text{Right}; k\right\rangle$ denotes any state in Eq.(2.37), and the sign is "$-$" when the right moving fermions are in R sector. The unoriented string is defined as the string theory invariant under the action of Ω. Its quantum states have identical left and right excitations: $\left|\text{Left}, \text{Right}; k\right\rangle$ with Left = $\overline{\text{Right}}$, and can be obtained by projecting out the asymmetric states in the whole spectrum (2.37) using the projection operator $\frac{1}{2}(1+\Omega)$.

2.4 Open $\mathcal{N} = (1,1)_2$ superstring and type I string

The initial classical action describing the open supersting is the same as for the closed superstring Eq.(2.28), which is reduced by lightcone gauge to the same SCFT action (2.29). However it is defined only on the upper-half z-plane. While the quantization of bosonic part is as explained in Sec.2.2, that of the fermionic sector proceeds in the same way as in Sec.2.3, but with right-moving sectors identified with the left-moving sector so the former do not appear explicitly in the result. Now it is each chiral fermion pair $\{\psi^{\tilde{\imath}}, \bar{\psi}^{\tilde{\imath}}\}$ that generates a Virasoro algebra representation $[0] + 2[\frac{1}{16}] + [\frac{1}{2}]$, with $[0] + [\frac{1}{2}]$ associated to the boundary conditions $\psi|_{\sigma=0} = \bar{\psi}|_{\sigma=0}$ and $\psi|_{\sigma=\pi} = \bar{\psi}|_{\sigma=\pi}$ (NS), and $[\frac{1}{16}]$ associated to $\psi|_{\sigma=0} = \bar{\psi}|_{\sigma=0}$ and $\psi|_{\sigma=\pi} = -\bar{\psi}|_{\sigma=\pi}$ (R). The mode expansions of chiral fermions is just like in the closed string case, but the holomorphic and the anti-holomorphic oscillators identified: $\psi^{\tilde{\imath}}_r = \bar{\psi}^{\tilde{\imath}}_r$. Again, NS boundary condition gives integer mode expansion, and R boundary condition gives half-integer mode expansion. All fermions should take the same boundary condition in order to preserve worldsheet supersymmetry, so the Hilbert space is decomposed into the NS/R sector giving rise to spacetime bosons and fermions respectively. Putting together the bosonic sector and the fermionic sector, a generic state in the whole Hilbert space is like

$$|X_{\text{op}}\rangle \otimes \begin{Bmatrix} |\text{NS}\rangle_\psi \\ |\text{R}\rangle_\psi \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{spacetime bosons} \\ \text{spacetime fermions} \end{Bmatrix}, \tag{2.42}$$

where $|X_{\text{op}}\rangle$ is an open bosonic string state as given in Eq.(2.25).

Unoriented open string

The worldsheet parity operator Ω acts on the open string coordinates in the pattern $\Omega : \sigma \to \pi - \sigma$. Its action on the oscillators are

$$\Omega\, \alpha^{\tilde{\imath}}_n\, \Omega^{-1} = (-1)^n \alpha^{\tilde{\imath}}_n\,; \quad \Omega\, \psi^{\tilde{\imath}}_r\, \Omega^{-1} = (-1)^r \psi^{\tilde{\imath}}_r\,, \tag{2.43}$$

which implies that its action on any physical state introduces a phase $(-1)^N$ where N is the total oscillator level. Also Ω acts non trivially on the ground states, which are

$$\begin{aligned} \mathrm{NS}: \quad & \Omega\,|0;\vec{k};ij\rangle = -i(\gamma_\Omega)_{i\,i'}(\gamma_\Omega^{-1})_{j'j}\,|0;\vec{k};j'i'\rangle, \\ \mathrm{R}: \quad & \Omega\,|S_\alpha;\vec{k};ij\rangle = (\gamma_\Omega)_{i\,i'}(\gamma_\Omega^{-1})_{j'j}\,|S_\alpha;\vec{k};j'i'\rangle, \end{aligned} \tag{2.44}$$

where in the first line the "0" is for indicating the NS sector ground state which has no oscillator, and in the second line S_α is for indicating the R sector ground state which is a spacetime spinor; also k is the transverse momentum as in Eq.(2.18). The matrix γ_Ω is a unitary, acting on the Chan-Paton factors and satisfying

$$\gamma_\Omega^T = \zeta\gamma_\Omega \quad \text{with} \quad \zeta = \pm 1\,. \tag{2.45}$$

These constructions guarantee $\Omega^2 = 1$ on the quantum level. The unoriented open string is supposed to be invariant under Ω and the spectrum can be obtained by acting the projection operator $\frac{1}{2}(1+\Omega)$ on the whole spectrum (2.42).

Type I string

The type I string is a model which consists of closed and open unoriented superstrings of worldsheet supersymmetry $\mathcal{N} = (1,1)_2$. The Hilbert space is constructed as follows:

Closed string sector: containing type IIB states specified in the end of Sec.2.3, subjected in addition to the worldsheet parity invariance Eq.(2.41).

Open string sector: containing openstring stats as in Eq.(2.42), supplemented in addition with GSO condition (same as in closed superstrings) and invariance under the action of Ω; for sake of good UV behavior, the Chan-Paton factors range from 1 to 32 and γ_Ω in Eq.(2.44) should be symmetric, which results in the $SO(32)$ gauge group in spacetime.

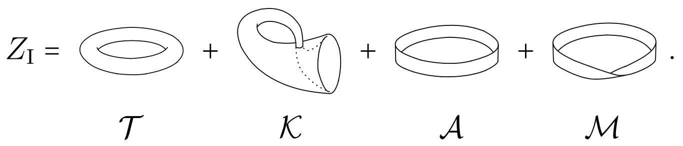
The type I theory has maximally 16 supercharges and the corresponding low energy effective theory in 10 dimensions is $\mathcal{N}_{10} = 1$ supergravity with $SO(32)$ gauge symmetry. The one-loop amplitude is a sum of closed string amplitude and open string amplitude, but with only unoriented string states circulating in the loop. To pick out only the unoriented string states, one needs only to insert the projection operator $\frac{1}{2}(1+\Omega)$ in the trace when computing conformal characters Eq.(2.5), i.e. $\mathrm{Tr}\big[\frac{1}{2}(1+\Omega)\,q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\big]$ for closed string and $\mathrm{Tr}\big[\frac{1}{2}(1+\Omega)\,q^{L_0-c/24}\big]$ for open string. More accurately, with the insertion of Ω, the conformal character for closed string is obtained by

$$\begin{aligned} &\mathrm{Tr}\,q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}} = (q\bar{q})^{h-\frac{c}{24}}\sum_{N,\bar{N}} D_N\bar{D}_{\bar{N}}q^N\bar{q}^{\bar{N}} \\ &\quad\longrightarrow\ \mathrm{Tr}\,\Omega\,q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} = \pm(q\bar{q})^{h-\frac{c}{24}}\sum_M D_M(q\bar{q})^M, \end{aligned} \tag{2.46}$$

where the + sign is for NS-NS sector, − for RR, and the nonzero contributions must come from left-right symmetry modes: $D_M = \bar{D}_M$, $c = \bar{c}$ and $h = \bar{h}$. In this case the Ω-insertion gives rise to the worldsheet topology of Klein bottle. Then for the open string, the insertion of Ω leads to

$$\mathrm{Tr}\, q^{L_0 - \frac{c}{24}} = q^{h - \frac{c}{24}} \sum_N D_N q^N \;\longrightarrow\; \mathrm{Tr}\, \Omega\, q^{L_0 - \frac{c}{24}} = q^{h - \frac{c}{24}} \sum_N (-1)^N D_N q^N, \quad \text{where} \quad q = e^{-\pi\tau_2}, \tag{2.47}$$

which leads to the worldsheet topology of Möbius strip. The result for the closed string sector turns out to be the sum of a torus amplitude $\mathcal{T}$ and a Klein bottle amplitude $\mathcal{K}$, while that for the open string sector is the sum of an annulus amplitude $\mathcal{A}$ and a Möbius strip amplitude $\mathcal{M}$:



(2.48)

The torus amplitude $\mathcal{T}$ is just half of the type II amplitude Eq.(2.39). The internal momenta is as Eq.(2.40) but there is no anti-symmetric tensor $B_{\alpha\beta}$, since the corresponding vertex operator is ruled out by orientifold projection. The Klein bottle amplitude is

$$\mathcal{K} = \frac{V_D}{2(2\pi)^D} \frac{1}{2} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \frac{1}{\eta^8} \sum_p q^{p^2} \sum_{a,b} (-1)^{a+b+ab} \frac{\theta[^a_b]^4}{2\eta^4}, \quad \text{where} \quad q = e^{-4\pi\tau_2}. \tag{2.49}$$

Here the sum over the internal momenta goes through those satisfying $p_L^I = p_R^I$, i.e. those as in Eq.(2.40) but with no $B_{\alpha\beta}$ and winding numbers. It is worth noticing that the Klein bottle amplitude is from the closed string sector and contains only contribution from left-right symmetric states. Therefore it has only contribution from NS-NS states ($a = 0$) and RR states ($a = 1$), which are all spacetime bosons. We note that the NS-NS sector contribution to $\mathcal{K}$ is positive while the RR sector contribution is negative. This assignment of phase guarantees that when switching to the transverse channel $\tilde{\mathcal{K}}$ (putting $\tau_2 = 1/2\ell$ and re-expressing the integrand in $\mathcal{K}$ in terms of $i\ell$ using modular transformations), the closed string modes propagating between the two crosscaps are physical. As a result, the sum of $\mathcal{T}$ and $\mathcal{K}$ symmetrizes the NS-NS sector but anti-symmetrizes the RR sector, and spacetime supersymmetry is preserved. The open string amplitudes are

$$\mathcal{A} = \frac{V_D}{2(2\pi)^D} \frac{N^2}{2} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \frac{1}{\eta^8} \sum_p q^{p^2} \sum_{a,b} (-1)^{a+b+ab} \frac{\theta[^a_b]^4}{2\eta^4}, \tag{2.50}$$

$$\mathcal{M} = \frac{V_D}{2(2\pi)^D} \frac{\zeta N}{2} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \frac{1}{\hat{\eta}^8} \sum_p q^{p^2} \sum_{a,b} (-1)^{a+b+ab} \frac{\hat{\theta}[^a_b]^4}{2\hat{\eta}^4}, \tag{2.51}$$

where $q = e^{-\pi\tau_2}$, and the hat on the functions in $\mathcal{M}$ means that terms of odd oscillator level changes sign following Eq.(2.47), which is first introduced in [32] (c.f. also Eq.(85) in [33]). Here N is

Chan-Paton multiplicity (not to be confused with oscillator level), and ζ the sign introduced in Eq.(2.45), which should take the value $N = 32$ and $\zeta = -1$ to make the total amplitude (2.48) UV finite. Here for simplicity we do not consider Wilson line deformation.

2.5 Heterotic superstring

The heterotic string has 10 left moving bosons with chiral fermion superpartners $\{X_L^\mu, \psi^\mu\}$, and 26 right-moving bosons $\{X_R^\mu, X_R^{\tilde I}\}$, where $\mu = 0, \ldots 9$ and $\tilde I = 10, \ldots, 25$. The spacetime is of dimension 10 labeled by μ, while the extra right-moving bosons $X_R^{\tilde I}$ are compactified on some 16 dimensional internal space. Imposing the lightcone gauge so that lightcone components ($\mu = 0, 1$) are eliminated, and we are left with an $\mathcal{N} = (1,0)_2$ SCFT of central charge $(c^{\text{het}}, \bar c^{\text{het}}) = (12, 24)$. The quantization of spacetime components $X^{\tilde\imath}$ and $\psi^{\tilde\imath}$ ($\tilde\imath = 2, \ldots, 9$) proceeds as in Sec.2.3. The 16 internal right-moving bosons $\{X_R^{\tilde I}\}$ have mode expansions

$$X_R^{\tilde I} = -i\,\frac{l_s^2}{2}\,p_R^{\tilde I}\ln\bar z \,+\, i\,\frac{l_s}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\frac{\bar\alpha_n^{\tilde I}}{\bar z^n}. \tag{2.52}$$

Here $p_R^{\tilde I}$ ($\tilde I = 10, \ldots, 25$) should be distinguished from p_R^I ($I = D, \ldots, 9$). In 10 dimensional spacetime, $p_R^{\tilde I}$ lies in an $E_8 \times E_8$ or $\mathrm{Spin}(32)/Z_2$ lattice, while upon toroidal compactification to lower dimensions, they can be subject to Wilson line deformation. A generic state is given by

$$\begin{array}{ccccc} \{X_L^{\tilde\imath}, X_R^{\tilde\imath}\} & & \{\psi^i\} & & \{X_R^{\tilde I}\} \\ \downarrow & & \downarrow & & \downarrow \\ |X_{\text{cls}}\rangle & \otimes & \left\{\begin{array}{c} |\text{NS}\rangle_\psi \\ |\text{R}\rangle_\psi \end{array}\right\} & \otimes & |N_{\text{int}}, q_R\rangle_{\text{int}} \,. \\ & & & & \| \\ & & & & \alpha_{-n_1}^{\tilde K_1}\cdots\alpha_{-n_l}^{\tilde K_l}{:}\exp[i\,p_R^{\tilde I}X_R^{\tilde I}(0)]{:}|0\rangle_{\text{int}} \end{array} \tag{2.53}$$

There is one graviton and one gravitino arising from the NS and the R sector respectively. Thus the maximally supersymmetric effective theory in 10 dimensions is the $\mathcal{N}_{10} = 1$ supergravity with $SO(32)$ or $E_8 \times E_8$ gauge symmetry. The total conformal character is obtained by assembling all the building blocks corresponding to each Hilbert space sector in Eq.(2.53):

$$\begin{aligned} \chi_{\text{het}}(\tau,\bar\tau) &= \chi_{k,p}^{X}(\tau,\bar\tau)\big[\chi_{\text{GSO}}^{\text{NS}}(\tau) + \chi_{\text{GSO}}^{\text{R}}(\tau)\big] \\ &= \left[\frac{q^{\frac{l_s^2}{4}(k^2+p_L^2)}\,\bar q^{\frac{l_s^2}{4}(k^2+p_R^2)}}{\eta^8\bar\eta^{24}}\right]_X \left[\sum_{a,b}(-1)^{a+b+ab}\frac{\theta[{}^a_b]^4}{2\eta^4}\right]_\psi . \end{aligned} \tag{2.54}$$

Here by p_R^2 we mean $p_R^I p_R^I + p_R^{\tilde{I}} p_R^{\tilde{I}}$. The index a indicates whether the corresponding term is from the NS sector ($a = 0$) or the R sector ($a = 1$). Thus a is the spacetime fermion number (recall that in type II it is $a + \bar{a}$). The sum over b implements the GSO projection. The one-loop vacuum amplitude is

$$\begin{aligned} Z_{\rm h} =& \frac{V_D}{2(2\pi)^D} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} \sum_{\substack{p_L, p_R \\ q_R}} \frac{q^{\frac{l_s^2}{4} p_L^2} \bar{q}^{\frac{l_s^2}{4} p_R^2}}{\eta^8 \bar{\eta}^{24}} \sum_{a,b} (-1)^{a+b+ab} \frac{\theta[^a_b]^4}{2\eta^4} \\ =& \frac{V_D}{2(2\pi)^D} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} \frac{\Gamma_{(d,16+d)}}{\eta^8 \bar{\eta}^{24}} \left(V_8 - S_8 \right). \end{aligned} \tag{2.55}$$

It shows that the spacetime supersymmetry is realized in the left-moving sector. The generic form of the internal momenta is [34]

$$\begin{aligned} p_{L,R}^I &= \left(m_\alpha - Q^{\tilde{I}} Y_\alpha^{\tilde{I}} - l_s^{-2} n^\beta B_{\alpha\beta} - \frac{1}{2} n^\beta Y_\alpha^{\tilde{I}} Y_\beta^{\tilde{I}} \right) e^{*\alpha I} \mp l_s^{-2}\, n^\alpha e_\alpha^I \,, \\ p_R^{\tilde{I}} &= \sqrt{2}\, l_s^{-1} \left(Q^{\tilde{I}} + n^\alpha Y_\alpha^{\tilde{I}} \right) \,, \quad \text{with} \ \ \{Q^{\tilde{I}}\} \text{ in Spin}(32)/Z_2 \text{ or } E_8 \times E_8 \text{ lattice,} \end{aligned} \tag{2.56}$$

where we have the same moduli fields as in Eq.(2.40) for the type II case, except that we have Wilson lines $\{Y_\alpha^{\tilde{I}}\}$ of the gauge group. At a generic point in the moduli space, the gauge group is $U(1)_L^d \times U(1)_R^{d+16}$, each $U(1)$ factor corresponding to the rotation around an internal circle. Enhanced gauge symmetry can be obtained when extra massless states emerge supplying non Cartan generators. We will come back to this point later. In particular when there is no Wilson line deformation $Y_\alpha^{\tilde{I}} = 0$, the lattice sum over $\{p_R^{\tilde{I}}\}$ decouples from the rest: $\Gamma_{(d,d+16)} = \Gamma_{(d,d)}\Gamma_{(0,16)}$, and the sum runs through the $E_8 \times E_8$ or $\text{Spin}(32)/Z_2$ lattice, yielding

$$\Gamma_{(0,16)} = \sum_{p_R^{\tilde{I}}} \bar{q}^{\frac{1}{4} p_R^{\tilde{I}} p_R^{\tilde{I}}} = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{\delta,\gamma} \theta[^\delta_\gamma]^8 \dfrac{1}{2} \sum_{\delta',\gamma'} \theta[^{\delta'}_{\gamma'}]^8, & E_8 \times E_8 \text{ lattice,} \\ \\ \dfrac{1}{2} \displaystyle\sum_{\delta,\gamma} \theta[^\delta_\gamma]^{16}, & \text{Spin}(32)/Z_2 \ \text{ lattice.} \end{cases} \tag{2.57}$$

Chapter 3

More about compactification

Last chapter was very brief about the compactifications of string theories. In this chapter we come back to this issue and discuss some more specific aspects encountered in our work. We will go further beyond toroidal compactification to consider orbifold and Calabi-Yau compactifications. From now on, we set $l_s = 1$ for simplicity of notation.

3.1 Lattice sum and worldsheet instanton

Upon toroidal compactifications, a lattice sum appears in the one loop amplitude as in Eqs (2.39) and (2.51), which takes the form

$$\Gamma_{(d_L,d_R)} = \sum_{p_L,p_R} q^{\frac{1}{4}p_L^2}\bar{q}^{\frac{1}{4}p_R^2}, \quad \text{or} \quad \Gamma_d = \sum_p q^{\frac{1}{2}p^2} \tag{3.1}$$

for closed and open strings respectively. For closed strings $(d_L, d_R) = (10-D, 10-D)$ for $\mathcal{N}_2 = (1,1)$ superstrings, and $(10-D, 26-D)$ for heterotic string. This piece of contribution to the one loop amplitude comes from worldsheet instanton effect, where the instanton numbers describe different ways that the worldsheet wraps the noncontractible cycles in the internal compact space. The simplest example is compactification on a circle, say let the 9-th spacetime dimension be compactified on a circle of radius R_9. Thus we have

$$p^9_{L,R} = \frac{m_9}{R_9} \mp n_9 R_9\,, \quad \text{where} \quad m_9,\, n_9 \in \mathbb{Z}\,, \tag{3.2}$$

with the integers m_9 and n_9 the KK excitation level and the winding number respectively, and the lattice sum in Eq.(3.1) running through all values of m_9 and n_9. Using the Poisson resummation on m_9 we rewrite the lattice sum into an instanton sum

$$\Gamma_{(1,1)} = \sum_{m_9,n_9} q^{\frac{1}{4}\left(\frac{m_9}{R_9}-n_9R_9\right)^2} \bar{q}^{\frac{1}{4}\left(\frac{m_9}{R_9}+n_9R_9\right)^2} = \frac{R_9}{\sqrt{\tau}_2} \sum_{\tilde{m}_9,n_9} \exp\left(-\frac{\pi R_9^2}{\tau_2}\left|\tilde{m}_9 - \tau n_9\right|^2\right). \tag{3.3}$$

The right hand side is an instanton sum because the path integral derivation (c.f. Sec.4.18 of [30]) reveals that each single term in the sum is from the configuration that the worldsheet, which has the topology of torus, has one of its homological cycle wrapping $\tilde{m}_9$ times $S^1(R_9)$, and the other homological cycle n_9 times. Taking the large volume limit $R_9 \to \infty$, we decompactify the 9-th dimension. All the nonzero instanton modes in Eq.(3.3) are exponentially suppressed, and we are left with $\frac{R_9}{\sqrt{\tau_2}}$ which is just the contribution of a noncompact direction to one-loop amplitude.

For open strings with only D9 branes discussed in the last chapter, which cannot wrap around $S^1(R_9)$, the internal momentum contains only Kaluza-Klein modes:

$$p^9 = \frac{m_9}{R_9}, \quad \text{where} \;\; m_9 \in \mathbb{Z}, \tag{3.4}$$

and the instanton sum goes like Eq.(3.3) but with $n_9 = 0$:

$$\Gamma_1 = \sum_{m_9} q^{\frac{1}{2}\frac{m_9^2}{R_9^2}} = \frac{R_9}{\sqrt{\tau}_2} \sum_{\tilde{m}_9} \exp\left(-\frac{\pi R_9^2}{\tau_2}\tilde{m}_9^2\right), \quad \text{where} \;\; e^{-\pi\tau_2}. \tag{3.5}$$

This is exactly the instanton sum for the point particle case, where $\tilde{m}_9$ is the winding number of the closed worldline around the Euclidean time circle.

Worldsheet instanton sum in more complicated cases, involving more compact dimensions and containing more moduli fields, can always be obtained by Poisson-ressumming all Kaluza-Klein excitation numbers in the lattice sum (3.1).

3.2 Enhanced gauge symmetry in toroidal compactification

Toroidally compactified heterotic string can have enhanced gauge symmetry at certain points in the moduli space. We start out with the mass spectrum, which is obtained by expanding the integrand in the one-loop amplitude Eq.(2.55) and reading off the powers of $q\bar{q}$:

$$M_{\text{het}}^2 = \frac{1}{2}\left(p_L^I p_L^I + p_R^I p_R^I + p_R^{\tilde{I}} p_R^{\tilde{I}}\right) + 2N + 2\bar{N} - 2, \tag{3.6}$$

where the internal momenta are as given in Eq.(2.56), N and $\bar{N}$ are left and right oscillator levels, and one has also the level-matching condition

$$p_L^I p_L^I + 4N = p_R^I p_R^I + p_R^{\tilde{I}} p_R^{\tilde{I}} + 4\bar{N} - 4. \tag{3.7}$$

Due to the fact that the ground state in the right-moving sector is tachyonic, certain oscillator ground states ($N = \bar{N} = 0$) of zero transverse momentum with nontrivial KK excitations and winding numbers can arrange the moduli to have

$$p_L^I p_L^I = 0, \quad p_R^I p_R^I + p_R^{\tilde{I}} p_R^{\tilde{I}} - 4 = 0, \tag{3.8}$$

so that the spacetime vector modes created by the vertex operators

$$V^i_{(p)}(z,\bar{z}) = \psi^i(z)\, \exp\Big[i\, p_R^I X_R^I(\bar{z}) + i\, p_R^{\tilde{I}} X_R^{\tilde{I}}(\bar{z})\Big] \tag{3.9}$$

are extra massless physical states in the heterotic string spectrum. There can be many of these states at one specific choice of moduli value, each state corresponding to different Kaluza-Klein and winding numbers. Also they come in pairs because when p_R satisfies Eq.(3.8) so does $-p_R$. These states lead to the enhancement of current algebra

$$i\,\bar{\partial}X^I(\bar{z})\, V^i_{(p)}(w,\bar{w}) = \frac{p_R^I}{\bar{z}-\bar{w}}\, V^i_{(p)}(w,\bar{w}), \quad i\,\bar{\partial}X^{\tilde{I}}(\bar{z})\, V^i_{(p)}(w,\bar{w}) = \frac{p_R^{\tilde{I}}}{\bar{z}-\bar{w}}\, V^i_{(p)}(w,\bar{w}), \tag{3.10}$$

with respect to the $U(1)$-currents carried by $i\,\bar{\partial}X^{I,\tilde{I}}$. Define the vectors $\vec{\alpha}_p := \left(p_R^I, p_R^{\tilde{I}}\right)$ and we can consider them as root vectors of some non-Abelian group. Thus the current algebra (3.10) shows that a larger gauge group arises, and the vector modes corresponding to the operators $i\,\psi^i(z)\,\bar{\partial}X^{I,\tilde{I}}(\bar{z})$ and Eq.(3.9) are gauge bosons of this group, because they are massless and transform in the adjoint representation. In particular the states associated to Eq.(3.9) are the non Cartan components of the group.

An obvious example is the $SU(2)$ enhanced symmetry when we compactify the heterotic string on a circle, say, again $S^1(R_9)$ with vanishing Wilson lines. Therefore the radius R_9 is the only modulus, and the internal momenta are as in Eq.(3.2). In this case when $R_9 = 1$, the internal momenta with $m_9 = n_9 = \pm 1$ satisfy the condition Eq.(3.8), where the two root vectors are simply $p_R^9 = \pm 2$. Thus the vertex operators $\frac{i}{2}\bar{\partial}X^9(\bar{z})$ and $\exp\big[\pm 2\, i\, X_R^{\tilde{I}}(\bar{z})\big]$ form an $SU(2)$ current algebra, and we obtain enhanced gauge symmetry $SU(2)$. Larger enhanced gauge symmetry can be achieved when we carry more internal moduli into play. In fact when the heterotic string is toroidally compactified down to D dimensions, the moduli can adjust to have $SO(52-2D)$ as the largest enhanced gauge symmetry. Actually fermionic construction of the heterotic string [35] can better reveal the possibilities of enhanced gauge symmetry.

For type II strings, since there is no tachyonic states in the spectrum, any massless state must have $p_L^2 = p_R^2 = 0$ upon toroidal compactification. In such cases no Kaluza-Klein or winding modes can become massless to supply the non Cartan components, so that we do not have gauge symmetry enhancement by any chance from perturbative effects. However certain orbifold compactifications can reverse the GSO projection, yielding a tachyonic ground state in type II superstrings. In such cases gauge symmetry enhancement can actually arise, for example it can be the case for Scherk-Schwarz reduction which we will discuss shortly, where an example giving rise to $SU(2)$ enhanced gauge symmetry is shown in [11].

3.3 Supersymmetry breaking by orbifold compactification

Maximally supersymmetric string theories contain too many supersymmetries for phenomenological application[1]. Orbifold compactification provides possibilities of breaking supersymmetry, where one compactifies the string theory on some internal manifold M having a discrete symmetry group G, and promotes G to gauge symmetry of the system. In this case we say that the theory is compactified on orbifold M/G. Cases of interest are those where the metric on M is flat so that the CFT is known and exact quantization is possible. The gauging of the orbifold group G can break explicitly or spontaneously supersymmetry.

Explicit breaking

By explicit supersymmetry breaking one simply seeks to discard part of the supercharges. More accurately, one choses the orbifold group such that some of the supercharges are invariant under its action, so that the other supercharges are eliminated when the orbifold group is promoted to gauge group. There can be many possibilities to achieve this goal but here we just take a simple example which suffices to illustrate the idea. We look at orbifold T^4/Z_2, where T^4 contains the directions 6 7 8 9, and the symmetry group Z_2 sends $X^{6,7,8,9}$ to $-X^{6,7,8,9}$, and the same thing for $\psi^{6,7,8,9}$ and $\bar{\psi}^{6,7,8,9}$. In this case the generator of Z_2 can be written as

$$g = \exp\left[i\pi(J^{67} + J^{89})\right]. \tag{3.11}$$

Here J^{**} are angular momentum operators in the internal space, which have integer eigenvalues for spacetime bosons and half-integer eigenvalues for spacetime fermions. Thus $J^{67} + J^{89}$ always has integer eigenvalues. The definition Eq.(3.11) shows that gauging the orbifold symmetry group amounts to eliminating states with odd eigenvalues of $J^{67} + J^{89}$. Therefore the gauging projects out half of the supercharges, explicitly breaking half of the supersymmery. It is worth noticing that modular invariance requires the introduction of twisted states in the Hilbert space, but in

[1] As is explained in the last chapter, we have 16 supercharges in type II strings and 8 in type I and heterotic strings, too much with respect to 4 supercharges for $\mathcal{N}_4 = 1$ or 8 supercharges for $\mathcal{N}_4 = 2$ models.

the current example, the twisted sector has the same amount of supersymmery as the untwisted sector. One step further, the orbifold $T^6/Z_2 \times Z_2$ reduces the amount of supercharge by 1/4.

The explicit partition function description of the spectrum and the computation one-loop amplitude is standard technical issue and is model-dependent. We are not intended to go through it in detail but will just take the known results from references.

Spontaneous breaking

Also, orbifold compactification can induce mass gaps between bosonic modes and fermonic modes, which spontaneously breaks supersymmetry. Still we take just an example to sketch out the idea. Consider Scherk-Schwarz reduction [36] on a circle $S^1(2R_9)/Z_2$ where the generator of Z_2 is

$$g_{\text{ss}} = (-1)^Q \delta. \tag{3.12}$$

Here δ is a shift by half of the R_9-circle $X^9 \to X^9 + \pi(2R_9)$, and Q is some conserved charge in spacetime which has different values for bosonic states and fermonic states. A most handy example is taking $Q = a$ the spacetime fermonic number in heterotic string. To see what happens in the spectrum when we gauge this symmetry, we write down a generic quantum state before gauging

$$\left| \frac{m_9}{2R_9}, 2n_9R_9 ; a ; \dots \right\rangle = e^{i(p_L^9 X_L^9 + p_R^9 X_R^9)} \left| 0 , 0 ; a ; \dots \right\rangle, \quad \text{with } \; p^9_{L,R} = \frac{m_9}{2R_9} \mp 2n_9R_9 , \tag{3.13}$$

the ellipses standing for irrelevant quantum numbers. To make connection to Eq.(2.53): the quanta m_9 and n_9 are in the $|X_{\text{cls}}\rangle$ part, and $a = 0$ corresponds to the NS sector, $a = 1$ the R sector. Therefore we have boson-fermion pairs of supersymmetry

$$\left| \frac{m_9}{2R_9}, 2n_9R_9 ; 0 ; \dots \right\rangle \xleftrightarrow{\text{SUSY}} \left| \frac{m_9}{2R_9}, 2n_9R_9 ; 1 ; \dots \right\rangle. \tag{3.14}$$

The orbifold action is

$$\begin{aligned} g_{\text{ss}} \left| \frac{m_9}{2R_9}, 2n_9R_9 ; a ; \dots \right\rangle &= \left[g_{\text{ss}} e^{i(p_L^9 X_L^9 + p_R^9 X_R^9)} g_{\text{ss}}^{-1} \right] g_{\text{ss}} \left| 0 , 0 ; a ; \dots \right\rangle \\ &= e^{i(p_L^9 X_L^9 + p_R^9 X_R^9)} e^{2\pi i R_9 (p_L^9 + p_R^9)} (-1)^a \left| 0 , 0 ; a ; \dots \right\rangle = (-1)^{a+m_9} \left| \frac{m_9}{2R_9}, 2n_9R_9 ; a ; \dots \right\rangle. \end{aligned} \tag{3.15}$$

Therefore the states invariant under g_{ss} in the bosonic and fermonic sector are respectively

$$\left| \frac{2m_9}{2R_9}, 2n_9R_9 ; 0 ; \dots \right\rangle \quad \text{and} \quad \left| \frac{2m_9+1}{2R_9}, 2n_9R_9 ; 1 ; \dots \right\rangle. \tag{3.16}$$

Given that the mass formula is $M^2 \propto p_L^2+p_R^2+\dots$ where the dots stand for the oscillator excitations, we see that a mass gap $\Delta M \sim R_9^{-1}$ is generated between bosons and fermions, so that that the relation (3.14) no longer exists. This corresponds to a spontaneous supersymmetry breaking at scale R_9^{-1}, and supersymmetry is restored at decompactification limit $R_9 \to \infty$. It should be mentioned that when gauging the symmetry (3.12), one needs to introduce twisted states following from the requirement of modular invariance. These are states with reversed GSO projection and are non supersymmetric neither. At the decompactification limit they become supermassive and decouple from the rest of the system and the restoration of supersymmetry is restored anyway. Technically when computing the partition function or the one-loop amplitude for Scherk-Schwarz reduction, we do not need to consider the untwisted sector and the twisted sector piece by piece. There is a simple way out. When Scherk-Schwarz reduction is performed on a circle of radius $2R$ with orbifold action (3.12), one simply replaces the lattice sum $\Gamma_{(1,1)}(2R)$ by a weighted instanton sum as follows

$$\Gamma_{(1,1)}(2R) \longrightarrow \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m},n} (-1)^{Q\tilde{m}+Q'n+\epsilon\,\tilde{m}n} \exp\left(-\frac{\pi R^2}{\tau_2}\left|\tilde{m}-\tau n\right|^2\right), \tag{3.17}$$

where Q' is the image of Q under the modular transformation $\tau \to -1/\tau$, and ϵ takes 0 or 1 in order to respect modular invariance. We will use this in Chpater 5.

3.4 Type II string on Calabi-Yau three-folds

Calabi-Yau (CY) compactifications of string theories are in the same spirit as explicit supersymmetry breaking upon dimension reduction, where the goal is to preserve a minimum amount of supercharges. On general grounds, when compactifying 10 dimensional superstring theories down to 4 dimensional spacetime, the conservation of minimum amount of supercharges requires the internal 6 dimensional space to have $SU(3)$-structure. Furthermore, demanding that the vacuum background does not break supersymmetry imposes stronger conditions. In case where no flux is turned on, the internal space must have $SU(3)$-holonomy. Compactification on CY 3-folds (CY$_3$'s) are the most studied cases fulfilling these criteria. Although exact quantization can rarely be achieved for CY$_3$ compactifications, except in some special cases for example at Gepner points [37], yet with technical devices provided by algebraic geometry, one can acquire rich explicit knowledge of the resulting low energy effective theory from the topological and geometric characteristics of the CY$_3$.

Topological information of the CY$_3$'s that string theories care most about is encoded in the cohomology groups $H^{p,q}$ ($p,q=0,1,2,3$), whose dimensions $h^{p,q}$ are referred to as Hodge numbers. For a generic nonsingular CY$_3$ the Hodge numbers and the message that they carry are

- $h^{10} = h^{01} = 0:$ no globally defined covariant 1-form in 6D;
- $h^{03} = h^{30} = 1:$ one unique holomorphic 3-form Ω;
- $h^{00} = h^{33} = 1:$ constant solution to the Laplacian equation;
- $h^{11} = h^{22}:$ independent Kähler form deformations;
- $h^{21} = h^{12}:$ independent complex structure deformations.

Other Hodge numbers not appearing in the above list can be deduced from the relation $h^{p,q} = h^{3-p,3-q} = h^{q,p}$, and $h^{p,0} = h^{3-p,0}$. Here the first two lines above are consequences of $SU(3)$-structure, and the third line is obvious given that a CY_3 has only one connected component. It is worth some more explanation to the last two lines. We know that any infinitesimal CY_3 metric deformation can be separated into Kähler form deformations, which are in 1-1 correspondence to independent harmonic $(1,1)$-forms, and complex structure deformations, in 1-1 correspondence to harmonic $(1,2)$-forms. Also there is a theorem stating that for a compact smooth manifold, each cohomology class contains one unique harmonic representative. Therefore with each harmonic $(1,1)$ and $(1,2)$-form representing unambiguously a class in $H^{1,1}$ and $H^{1,2}$, there are in all h^{11} independent Kähler form deformations and h^{12} independent complex structure deformations. Let $\mathcal{M}_{\mathrm{K}}$ be the Kähler moduli space spanned by all infinitesimal Kähler form deformations, and in the same way we have $\mathcal{M}_{\mathrm{C}}$ the complex structure moduli space. They can be regarded as manifolds with local coordinates the linear combination coefficients of harmonic $(1,1)$ or $(1,2)$-forms, where $\dim\mathcal{M}_{\mathrm{K}} = h^{11}$ and $\dim\mathcal{M}_{\mathrm{C}} = h^{12}$. These two spaces are Kähler manifolds in their own right, whose geometry can be in principle worked out precisely once the CY_3 is known.

Upon CY_3 reduction of string theories, the resulting effective supergravity is determined at tree level by the geometric and topological aspects described above. Basically, massless fields in 4D are the expansion coefficients of 10D fields against the harmonic forms on CY_3, and harmonic forms are in 1-1 correspondence to cohomology classes. Therefore the cohomology groups indicate the types and the numbers of the 4D massless fields, as well as the structure of the effective supergravity moduli space. On the other hand the cohomology groups also control the structure of the geometric moduli space. Thus the physical moduli space of the effective supergravity can be related to some extent to the geometric moduli space of the CY_3. Cases of interest are those where the effective supergravity moduli space is protected by supersymmetry from quantum corrections, or where its quantum corrections are significantly suppressed. Then part of the geometric moduli space itself becomes part of the physical moduli space, and is just the exact quantum moduli space. Thus the related physics is completely dictated by CY_3 geometry.

This is just the case for CY_3 reduction of type II strings that we have been working on. The $SU(3)$-holonomy on the CY_3 preserves in 4D vacuum a quarter of supersymmetries in 10D. Thus we have at tree level $\mathcal{N}_4 = 2$ supergravity with Abelian gauge group as low energy effective theory. We will not enter into the details of the field reduction (c.f. for example, Sec.9.11 in [30]), but only highlight the essential aspects. As is stated, the 4D massless fields arise from reducing 10D fields against all harmonic forms living in the CY_3. For both type IIA and IIB strings, the reduction always yields:

- **One gravitational multiplet**: from field reduction against the $(0,0)$ $(3,0)$ and $(0,3)$-forms;
- **One universal hypermultiplet**: same reduction as above, containing the 4D dilaton.

In addition, for type IIA string, we have

- ◇ h^{11} **vector multiplets**: from field reduction against $(1,1)$-forms, whose scalar components are Kähler moduli of CY_3,
- ◇ h^{12} **hypermultiplets**: from field reduction against $(1,2)$ and $(2,1)$-forms, whose scalar components are complex structure moduli of CY_3,

while for type IIB string, we have instead

- ◇ h^{12} **vector multiplets**: from field reduction against $(1,2)$ and $(2,1)$-forms, whose scalar components are complex structure moduli of CY_3,
- ◇ h^{11} **hypermultiplets**: from field reduction against $(1,1)$-forms, whose scalar components are Kähler moduli of CY_3,

The CY_3 reduction of type IIA models and type IIB models are conjectured to be equivalent, mapped into each other by mirror symmetry. That is, type IIA string compactified on a CY_3 M of Hodge numbers $h^{11}(M)$ and $h^{12}(M)$ is the same model as type IIB string compactified on the mirror CY_3 W of Hodge numbers $h^{11}(W) = h^{12}(M)$ and $h^{12}(W) = h^{11}(M)$. However, it is by no means trivial to construct mirror CY_3's and to perform quantitative computations on both sides to check the validity of the conjecture.

In the rest of the section, we stay on the type IIB side, since the IIB picture displays more interesting features. Let all complex scalars in the vector multiplets be z^I with $I = 1, \ldots, h^{12}$, spanning the space $\mathcal{M}_{\mathrm{V}}$; also let all real scalars in hypermultiplets be q^Λ with $\Lambda = 1, \ldots, 4\times(h^{11}+1)$

including the universal hypermultiplet, living in the space $\mathcal{M}_{\mathrm{H}}$. By virtue of $\mathcal{N}_4 = 2$ supergravity, $\mathcal{M}_{\mathrm{V}}$ is a special Kähler manifold of complex dimension h^{12} and $\mathcal{M}_{\mathrm{H}}$ a quaternionic manifold of quaternionic dimension $h^{11}+1$; they are two factorized components of the whole moduli space: $\mathcal{M}_{\mathrm{tot}} = \mathcal{M}_{\mathrm{V}} \times \mathcal{M}_{\mathrm{H}}$. For the vector fields, let them be V^A and their field strength be $G^A = dV^A$, with $A = 0, \dots, h^{12}$ including the graviphoton. Thus the resulting $\mathcal{N}_4 = 2$ supergravity action reads

$$
\begin{aligned}
S_{\text{IIB/CY}_3} = & \frac{1}{\kappa^2_{(4)}} \int_{\mathcal{S}} d^4x \sqrt{-g} \Big[\frac{1}{2}\mathcal{R} - g_{I\bar{J}}\, \partial z^I \partial \bar{z}^J - h_{\Lambda\Sigma}\, \partial q^\Lambda \partial q^\Sigma \\
& + \frac{1}{2}\mathrm{Im}\mathcal{N}(z,\bar{z})_{AB}\, G^A\, G^B + \frac{1}{2}\mathrm{Re}\mathcal{N}(z,\bar{z})_{AB}\, G^A \, {\star G^B} + (\text{fermions}) \Big].
\end{aligned} \tag{3.18}
$$

Here in the sector containing vector multiplets and gravitational multiplet, $g_{I\bar{J}} = g_{I\bar{J}}(z)$ is the special Kähler metric on $\mathcal{M}_{\mathrm{V}}$, and we also have the gauge kinetic matrix $\mathcal{N}_{AB} = \mathcal{N}_{AB}(z,\bar{z})$. In the hypermultiplet sector $h_{\Lambda\Sigma} = h_{\Lambda\Sigma}(q)$ the quaternionic metric on $\mathcal{M}_{\mathrm{H}}$.

The vector multiplet sector contains more explicit information. The fact that the dilaton lives in a hypermultiplet excludes all string loop and spacetime instanton corrections from $\mathcal{M}_{\mathrm{V}}$. Also given that the scalars z^I are identified with complex structure moduli of the CY_3, the moduli space $\mathcal{M}_{\mathrm{V}}$ receives no worldsheet instanton correction[2], since the latter only depends on Kähler moduli. This makes the vector moduli space $\mathcal{M}_{\mathrm{V}}$ exact at tree level and is identified with the complex structure moduli space $\mathcal{M}_{\mathrm{C}}$. Thus inheriting from $\mathcal{M}_{\mathrm{C}}$, we have for the physical moduli space $\mathcal{M}_{\mathrm{V}}$ the Kähler potential

$$
\mathcal{K} = -\ln\left(i \int \Omega \wedge \bar{\Omega} \right) \tag{3.19}
$$

where Ω is the unique holomorphic 3-form on the CY_3. Let $(\mathcal{A}^A, \mathcal{B}_A)$, $A = 0, 1, ..., h^{12}$ be a symplectic basis of 3-cycles in the CY manifold, which satisfies

$$
\langle \mathcal{A}^A | \mathcal{B}_B \rangle = -\langle \mathcal{B}_B | \mathcal{A}^A \rangle = \delta^A_{\ B}, \quad \langle \mathcal{A}^A | \mathcal{A}^B \rangle = \langle \mathcal{B}_A | \mathcal{B}_B \rangle = 0. \tag{3.20}
$$

Without loss of generality, we can call $\mathcal{A}^A$ electric cycles and $\mathcal{B}_A$ magnetic cycles. Then we take the dual cohomology basis (α_A, β^A), such that

$$
\int_{\mathcal{A}^B} \alpha_A = \delta_{AB}\,, \qquad \int_{\mathcal{B}_B} \beta^A = \delta^{AB}\ . \tag{3.21}
$$

We expand the holomorphic 3-form against this basis, where arise the electric periods $\{X^A\}$ and magnetic periods $\{F_A\}$

$$
\Omega = X^A \alpha_A - F_A \beta^A\,. \tag{3.22}
$$

[2]In the type IIA picture, the vector moduli space is also free of string loop and spacetime instanton corrections, but it is corrected by worldsheet instanton effects, since the scalars in vector multiplets are Kähler moduli.

Here F_A should not be confused with the field strength F^A appearing in the supergravity action Eq.(3.18). The periods $\{X^A\}$ can be used locally as projective coordinates on $\mathcal{M}_{\rm V}$, and in the vicinity where $X^0 \neq 0$, we can use the following coordinates

$$t^I = \frac{X^I}{X^0} \quad \text{for} \quad I = 1, \dots, h^{12}, \tag{3.23}$$

which are called the special coordinates. They can replace z^I in the action (3.18) as scalar components of vector multiplets. The magnetic periods which are functions on $\mathcal{M}_{\rm V}$ should depend on $\{X^A\}$. It can be shown [38] that locally F_A can be obtained by taking the derivative of some function $\mathcal{F}(X)$ with respect to X_A:

$$F_A(X) = \frac{\partial}{\partial X^A}\mathcal{F}(X) = \partial_A \mathcal{F}(X), \tag{3.24}$$

and $\mathcal{F}$ is called the prepotential of the special Kähler manifold. Inserting Eq.(3.22) into Eq.(3.19), we have for the Kähler potential

$$\mathcal{K} = -\ln\left[i\left(\bar{X}^A F_A - X^A \bar{F}_A\right)\right]. \tag{3.25}$$

The gauge kinetic matrix can be computed by

$$\mathcal{N}_{AB} = \bar{\mathcal{F}}_{AB} + \frac{2i\,\mathrm{Im}\mathcal{F}_{AC}X^C\,\mathrm{Im}\mathcal{F}_{BD}X^D}{\mathrm{Im}\mathcal{F}_{EF}X^E X^F}, \tag{3.26}$$

where $\mathcal{F}_{AB} := \partial_A\partial_B\mathcal{F}$.

The sector of hypermultiplets is less clear. At tree level $\mathcal{M}_{\rm H}$ is a direct product of a sector containing Kähler moduli and a sector containing only the universal hypermultiplet. However it is further corrected by perturbative and non-perturbative quantum effects, and all that we know on general grounds is that the corrected $\mathcal{M}_{\rm H}$ is a quaternionic manifold.

Chapter 4

A glimpse to non-perturbative spectra

The end of the last century saw the emergence of various conjectures of string theory dualities and the cumulation of their supporting evidences. Up to now, all known string theories are interconnected by duality maps, and this strongly suggests the existence of a more profound theory, where each string theory arises as its different weak-coupling extremities. Although we are far from figuring out this underlying theory in all exactitude, the new vision has by all means greatly deepened our knowledge of string theory. One most important insight is that each string theory has a whole plentitude of non-perturbative spectrum that perturbative quantization is not able to reveal. A great amount of work has been devoted to the study of non-perturbative aspect of string theories. It is not only for need of supporting evidence to the duality conjectures, but also for exploring the phenomenological consequences that the non-perturbative effects entail. Often duality maps are used as tools for discovering non-perturbative effects.

The discussion in previous chapters ignores all non-perturbative effects, which we make up for here in this chapter. Certainly our discussion covers only the non-perturbative effects used in our work, which is merely a tiny tip of the whole panorama of the non-perturbative realm. The devices that we use to indicate the existence of non-perturbative effects are the S-dualities between string theories and also the singularities in the moduli spaces.

The S-duality concerns two weakly coupled theories where one is deep in the strong coupling regime of the other, with some duality map sending the two theories into each other. Thus the perturbative spectrum on one side turns out to be the non-perturbative spectrum on the other side. Among all the duality maps conjectured for string theories, the S-dualities include the self-duality of type IIB string, the duality connecting T^4-compactification of heterotic string and $K3$-compactification of type II string, and also the duality connecting type I string and heterotic string, both of $SO(32)$ gauge group. We will use the type I/heterotic duality [39] to tell about non-perturbative effects in the type I theory from the heterotic theory in Chapter 7.

Singularities in moduli spaces can indicate non-perturbative effects. In particular, here we care about the logarithmic singularities, which can be the signature of light states that are wrongly integrated out from the Wilsonian effective action. In order that such prediction is viable, the moduli space should be exact at quantum level. Generically, quantum corrections can modify the singularity of the moduli space. Thus extra massless states predicted by the quantum moduli space singularities are generically different from those predicted by classical moduli space singularities. A famous field theory example can be found in [40], where the author shows that instanton effects in $\mathcal{N}=1$ SQCD can correct the moduli space singularities and the resulting light states. For this reason, we study type II strings compactified on CY_3's, where the vector multiplet moduli space $\mathcal{M}_{\mathrm{V}}$ develops local singularity when certain sort of extremal transition undergoes in the CY_3. As explained in Sec. 3.4, the vector multiplet moduli space on the type IIB side at tree level is the exact quantum moduli space. For this reason we can rely on its singularities for prediction of non-perturbative states.

The two non-perturbative effect indicators mentioned above are not string theory peculiarities, but have already been well explored in field theories. It is worth mentioning Seiberg and Witten's work [41], which greatly inspired works on non-perturbative effects in string theories. In that work both S-duality and moduli space singularity are exploited, leading to the discovery of non-perturbative states. The model considered is pure $\mathcal{N}=2$ super Yang-Mills theory which has asymptotic freedom. Through an S-duality transformation one switch to the magnetic representation, where the theory becomes weakly coupled at IR. In the IR regime, logarithmic singularities are uncovered, whose monodromies imply the arising of massless magnetic monopoles or dyonic states.

4.1 D1-brane states in type I string

The conjecture of the S-duality relating the $SO(32)$ heterotic string and the $SO(32)$ type I string is motivated by the observation that the low energy effective field theory on the heterotic side can be mapped to the type I side and vice versa by suitable field redefinition. In 10D in Einstein frame, the bosonic effective supergravity action from the type I side and the heterotic side are respectively

$$\begin{aligned} S_{\mathrm{I}} &= \int d^{10}x\,\sqrt{-g}\Big[R-\frac{1}{2}\big(\nabla\phi_{\mathrm{I}}\big)^2-\frac{1}{4}e^{-\frac{\phi_{\mathrm{I}}}{2}}F^2-\frac{1}{12}e^{-\phi_{\mathrm{I}}}H^2\Big],\\ S_{\mathrm{h}} &= \int d^{10}x\,\sqrt{-g}\Big[R-\frac{1}{2}\big(\nabla\phi_{\mathrm{h}}\big)^2-\frac{1}{4}e^{\frac{\phi_{\mathrm{h}}}{2}}F^2-\frac{1}{12}e^{\phi_{\mathrm{h}}}H^2\Big], \end{aligned} \tag{4.1}$$

where $g_{\mu\nu}$ is the spacetime metric in the Einstein frame; $\phi_{\mathrm{I,h}}$ is the dilaton on the type I and the heterotic side; H is the field strength of 2-forms: on the heterotic side, $H_3 = dB_2$ with $B_{\mu\nu}$ the NS-NS anti-symmetric tensor, and on the type I side $H_3 = dC_2$ with $C_{\mu\nu}$ the RR 2-form; also we have $F = dA$ the field strength of $SO(32)$ gauge bosons. We observe that the two actions are sent into each other by the following map

$$g^{\mathrm{h}}_{\mu\nu} = g^{\mathrm{I}}_{\mu\nu}, \quad \phi_{\mathrm{h}} = -\phi_{\mathrm{I}}, \quad B_{\mu\nu} = C_{\mu\nu}, \quad A^{\mathrm{h}}_{\mu} = A^{\mathrm{I}}_{\mu}. \tag{4.2}$$

We emphasize that the metric here is in Einstein frame. Since the string coupling constant in 10D is the exponential of the dilaton $\lambda_{\mathrm{h,I}} = \exp\left(\phi_{\mathrm{I,h}}\right)$, the change in the sign of dilaton in the above map shows that $\lambda_{\mathrm{h}} = 1/\lambda_{\mathrm{I}}$ under the duality map so that it is an S-duality.

One pertinent consistency check of this S-duality arises from the identification of $B_{\mu\nu} = C_{\mu\nu}$ [42]. On the heterotic side, the perturbative F-string is electrically coupled to $B_{\mu\nu}$, while on the type I side the object electrically coupled to $C_{\mu\nu}$ is the non-perturbative D1-brane. This suggests that the heterotic F-string is just the D1-brane in the type I picture. In the weak coupling regime of type I string, one can examine the quantum fluctuations of D1-branes by considering two types of open strings: those with both ends attached to the D1-brane, and those with one end on D1 and the other on D9. The D1-D1 string gives rise to 8 bosonic states and 8 chiral fermionic states, while the D1-D9 string gives rise to 32 chiral fermionic states, whose chirality is opposite to the 8 chiral fermions from the D1-D1 sector. In all, the quantum fluctuations on the D1-brane match perfectly the field content on the heterotic F-string world sheet in the light cone gauge. This confirms the identification of the D1-brane in type I string with the F-string in the dual heterotic string.

Upon toroidal compactifications down to D-dimensional spacetime, the duality map between moduli fields can be derived from Eq.(4.2). In the most generic case the compactification gives rise to moduli fields including

- The D-dimensional dilaton $\phi^{(D)}_{\mathrm{h,I}} = \phi^{(10)}_{\mathrm{h,I}} - \frac{1}{2}\ln \hat{V}^{(d)}_{\mathrm{h,I}}$, with $\hat{V}^{(d)}_{\mathrm{h,I}}$ the internal space volume in string frame;
- The internal metric $\hat{g}^{\mathrm{h,I}}_{\alpha\beta}$ in string frame, so that $\hat{V}^{(d)} = \sqrt{\det \hat{g}}$;
- The internal NS-NS field $B_{\alpha\beta}$ and RR 2-form $C_{\alpha\beta}$;
- The Wilson lines of the $SO(32)$ group $Y^{\tilde{I}}_{(\mathrm{h,I})\alpha}$, with $\tilde{I}$ label of Cartan components.

Here we hat the quantities measured in string frame, and will always adopt this convention whenever we need to specify the reference frame. The duality map for these moduli fields is therefore [43]

$$
\begin{aligned}
&\hat{g}^{\mathrm{h}}_{\alpha\beta} = \lambda_{\mathrm{I}}^{-1}\,\hat{g}^{\mathrm{I}}_{\alpha\beta} = \left(\hat{V}^{(d)}_{\mathrm{h,I}}\right)^{-1/4} e^{-\phi^{(D)}_{\mathrm{I}}}\,\hat{g}^{\mathrm{I}}_{\alpha\beta}\,,\\
&\phi^{(D)}_{\mathrm{h}} = -\frac{D-6}{4}\,\phi^{(D)}_{\mathrm{I}} - \frac{D-2}{8}\ln\hat{V}^{(d)}_{\mathrm{I}},\\
&B_{\alpha\beta} = C_{\alpha\beta}\,,\quad Y^{\tilde{I}}_{\mathrm{h}\,\alpha} = Y^{\tilde{I}}_{\mathrm{I}\,\alpha}\,,
\end{aligned}
\tag{4.3}
$$

while Eq.(4.2) always holds for the 10D dilaton and for the spacetime fields[1]. The inverse of the above map is obtained by exchanging the subscripts h $\leftrightarrow$ I. Note that Eq.(4.3) is no longer S-duality for spacetime dimension $D \leq 6$, where the dilaton sign is kept. In 6D, the dilaton is exchanged with the internal volume while below 6D, weakly coupled type I string is mapped into weakly coupled heterotic string and strongly coupled to strongly coupled. We stress that D1-brane states on the type I side are BPS states whose masses are protected by supersymmetry. Therefore non-perturbative D1-brane spectrum in type I at weak (strong) coupling, obtained at $D < 6$ ($D > 6$) from the weakly coupled heterotic dual, still holds at strong (weak) coupling.

4.2 Extremal transition I: singular nodes

Topology change

We consider type IIA string compactified on CY_3 denoted by M, and suppose that mirror symmetry maps it into type IIB string compactified on the mirror CY_3 denoted by W. We consider the type of extremal transition[2] where 2-spheres in M shrink to separated nodes so that M takes a singular configuration $\check{M}$, and then the nodes are deformed into 3-spheres so that $\check{M}$ is desingularized into M'. In such cases the singular nodes in $\check{M}$ present conical structure of basis $S^2 \times S^3$ [45], which justifies the name conifold and therefore such extremal transition is called conifold transition. To be more precise, at conifold transition, let there be R 2-spheres shrinking to zero size in the CY_3 M, to arrive at the conifold configuration $\check{M}$ containing R singular nodes. We suppose that the R shrinking 2-spheres span an S-dimensional subspace of the homology group $H_{1,1}(M)$, or stated in another way, they are subject to $R - S$ constraints from homology relations. When $R - S > 0$,

[1]The map for the dilaton had been discussed independently earlier in [44].

[2]The extremal transition of CY_3's is a sort of topology change relating two distinct CY_3's M and M' of different Hodge numbers through a singular configuration $\check{M}$. In such transition, one obtains $\check{M}$ from M through a birational contraction (2-spheres shrinking), and by deforming singularities in $\check{M}$ into 3-spheres one obtains M'. See [46] and the Ref.[49] therein.

we can deform the singular nodes into 3-spheres to obtain a nonsingular CY_3 M' topologically different from M, where these R 3-spheres span an $R-S$ dimensional subspace in the $H_{1,2}(M')$ homology group[3]. Therefore at the conifold transition from M to M', the $H_{1,1}$ homology group loses an S-dimensional subspace while the $H_{1,2}$ group earns $R-S$ new independent components, so that the change in Hodge numbers is

$$h_{11}(M') = h_{11}(M) - S, \quad h_{12}(M') = h_{12}(M) + R - S. \tag{4.4}$$

An intuitive way to understand this Hodge number change is shown in Fig.4.1 which is an illustrated example with $R = 5$ and $S = 2$.

In the same way we describe this transition on the type IIB side to have W and W' connected by the conifold $\check{W}$, but now we see R 3-spheres subject to $R-S$ homology relations shrinking in W, to produce $\check{W}$ and then the nodes are blown up into R 2-spheres to have W', which span an $R-S$ subspace of $H_{1,1}(W')$. Thus the change in Hodge numbers is

$$h_{12}(W') = h_{12}(W) - S, \quad h_{11}(W') = h_{11}(W) + R - S. \tag{4.5}$$

Conifold transition can be described very explicitly. We have for example the famous model with quintic in type IIB picture [45], where $W = [4\|5]$ and $W' = \left[\begin{smallmatrix}4\\1\end{smallmatrix}\middle\|\begin{smallmatrix}41\\11\end{smallmatrix}\right]$.

It should be emphasized that the two sides of the transition do not necessarily exist simultaneously. In type IIA picture, when all of the shrinking 2-spheres are homologically independent, i.e. $R = S$, the transition stops at $\check{M}$ and the branch of M' cannot be obtained. In the opposite direction, when all the R 3-spheres in M' arising from deforming singular nodes are homologically independent, then the R nodes cannot be blown up into 2-spheres to obtain M. The same observation holds for type IIB models. In our discussion of low energy effective theory, we always suppose that the branch M or W is available. As far as our current knowledge can reach, we need to sit in this branch to know how to write down the effective gauge theory. It is because with the shrinking of 2-spheres in M (3-spheres in W), states from non-perturbative D-branes (D2 in M and D3 in W) wrapping these shrinking spheres become light and should be re-included in the low energy effective theory.

Non-perturbative black hole states

To bring into light these non-perturbative effects from D-branes, we use the type IIB picture where the vector multiplet moduli space $\mathcal{M}_V$ exact at tree level, and let the emergence of the light

[3]It is always possible to choose the complex structure such that the 3-spheres arising from deformation lie in the $H_{1,2}$ group.

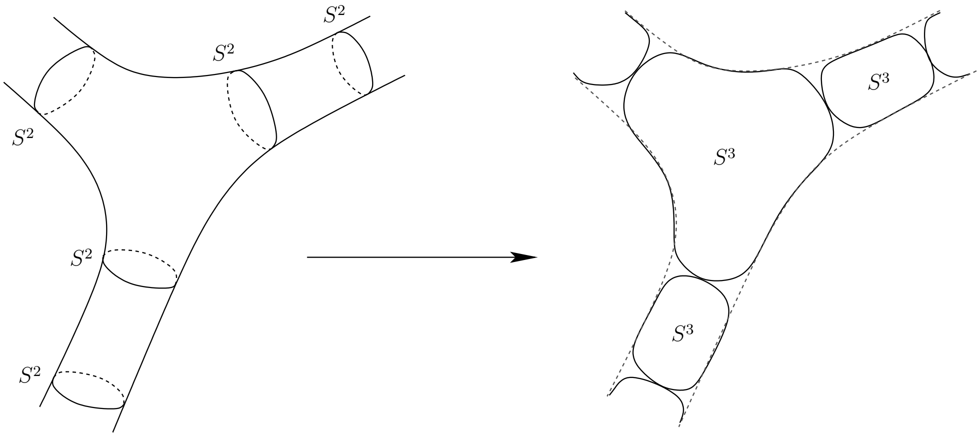


Figure 4.1: An intuitive illustration to understand the Hodge number change Eq.(4.4). It shows the situation where $R = 5$ 2-spheres, which span a subspace of the $H_{1,1}$ homology group of dimension $S = 2$, shrink to zero size. This gives birth to $R - S = 3$ new 3-spheres which are obviously homologically independent, and this leads to the result that h_{12} should increase by $R - S = 3$ when deforming the nodes into 3-spheres.

non-perturbative states be suggested by logarithmic singularities in it. Let the shrinking 2-spheres be $\{\mathcal{C}^{\hat{a}}\}$ $(\hat{a} = 1, \ldots, R)$, and we arrange the complex structure so that these shrinking 2-spheres can be expanded against the basis of electric cycles $\mathcal{C}^{\hat{a}} = n^{\hat{a}}_A \mathcal{A}^A$, where the matrix $n^{\hat{a}}_A$ is of rank S. Here we use the notation introduced in Eq.(3.20). Thus the vanishing locus of the sphere $\mathcal{C}^{\hat{a}}$ is $n^{\hat{a}}_A X^A = 0$, which is of co-dimension 1. We will refer to the intersection of all these R vanishing loci the conifold locus. By requirement of monodromy [47]

$$\mathcal{E} \longrightarrow \mathcal{E} + \mathcal{C}^{\hat{a}} \langle \mathcal{C}^{\hat{a}} | \mathcal{E} \rangle, \quad \text{(no sum over } a\text{)}, \tag{4.6}$$

for any 3-cycle $\mathcal{E}$ upon a transportation in the moduli space around the vanishing locus of $\mathcal{C}^{\hat{a}}$. This determines the singular piece in the magnetic periods n

$$F_A = \frac{1}{2\pi i} \sum_{\hat{a}} n^{\hat{a}}_A n^{\hat{a}}_B X^B \ln\left(n^{\hat{a}}_C X^C\right) + (\text{reg.}), \tag{4.7}$$

and inserting this into Eq.(3.26) we have that the gauge kinetic matrix has the singular behavior

$$\bar{\mathcal{N}}_{AB} = \frac{1}{2\pi i} \sum_{\hat{a}} n^{\hat{a}}_A n^{\hat{a}}_B \ln\left(n^{\hat{a}}_C X^C\right) + (\text{reg.}). \tag{4.8}$$

Due to the fact that the gauge coupling constant g_c is given by $g_c^{-2} \sim -\mathrm{Im}\mathcal{N}_{AB}$, as is displayed in Eq.(3.18), there appear in the gauge coupling constants the logarithmic singularities as $g_c^{-2} \sim -\ln\left|n^{\hat{a}}_C X^C\right|$. Here we especially display the minus sign to show that the gauge theory become weakly coupled when approaching the conifold locus. This predicts the emergence of light states since such logarithmic singularity arises exactly when states of mass $M \sim \left|n^{\hat{a}}_C X^C\right|$ are integrated out from the action (3.18). Again we emphasize that this prediction is credible because the moduli space is exact. In [20] it is postulated that these are R BPS black hole states becoming massless at the conifold locus, which are described by R hypermultiplets charged under $U(1)^S$. Since the gauge vectors are obtained from dimensional reduction of RR 4-forms, which are electrically coupled to D3-branes, the charge black hole states should arise from the D3-branes wrapping the vanishing 3-spheres.

This postulate is further supported by the analysis of Nambu-Goto action of the D3-brane. Upon reduction of the Nambu-Goto action on the supersymmetric special Lagrangian cycle represented by the vanishing 3-sphere $\mathcal{C}^{\hat{a}}$, one obtains in the 4D spacetime the action describing a point particle, which is in fact a black hole, of mass $M^a \sim \left|n^{\hat{a}}_C X^C\right|$, and of electric charges $n^{\hat{a}}_A$. Therefore we have a BPS black hole state of mass exactly as predicted by moduli space singularity. To see that the gauge group is $U(1)^S$, we chose the basis such that the matrix n^a_A takes the form $n^{\hat{a}}_i \neq 0$ $(i = 1, \ldots, S)$ and $n^{\hat{a}}_{A>S} = 0$. Thus the a-th hypermultiplet has charge $n^{\hat{a}}_i$ under the $U(1)$ associated with the vector multiplet containing X^i. In this basis the conifold locus is situated at $X^i = 0$ $(i = 1, \ldots, S)$.

Effective gauge theory

Now by "integrating in" the BPS black hole states in the Wilsonian effective action, we cure the IR singularities in the effective gauge theory. The conifold transition $M \to \check{M} \to M'$ matches perfectly the Coulomb-Higgs phase transition in this repaired nonsingular effective gauge theory. Here we illustrate this qualitatively in the field theory limit, while analysis based on supergravity will be carried out in Chapter 8.

Following from the fact that each of these black holes is charged under $U(1)^S$ associated to the scalars X^i with charge $n^{\hat{a}}_i$, the field theory Lagrangian in the $\mathcal{N}_4 = 1$ superspace language is

$$\begin{aligned} L_{\rm con} = \frac{1}{16\pi}\mathrm{Im}&\left[\int d^4\theta\, \Phi^{i\dagger}\exp(V^i)\,\Phi^i + \frac{1}{2}\int d^2\theta\, W^{i\alpha}W^i{}_\alpha\right] \\ &+ \int d^4\theta\left[H^{\hat{a}\dagger}\exp(2\,n^{\hat{a}}_i V^i)H^{\hat{a}} + \tilde{H}^{\hat{a}}\exp(-2\,n^{\hat{a}}_i V^i)\tilde{H}^{\hat{a}\dagger}\right] \\ &+ \sqrt{2}\,n^{\hat{a}}_i\left(\int d^2\theta\,\tilde{H}^{\hat{a}}\Phi^i H^{\hat{a}} + \text{h.c.}\right), \quad \text{summing over all repeated indices.} \end{aligned} \tag{4.9}$$

Here the superfields (Φ^i, V^i), each pair consisting of a chiral superfield and an $\mathcal{N}_4 = 1$ vector super field, constitute the $U(1)^S$ $\mathcal{N}_4 = 2$ vectormultiplets, while the doublets of chiral superfields $(H^{\hat{a}}, \tilde{H}^{\hat{a}})$ $(\hat{a} = 1, \ldots, R)$ constitute the charged black hole hypermultiplets. The complex scalars X^i are contained in Φ^i, and we denote the scalars in the R charged black hole hypermultiplets by

$\mathscr{H}^{\hat{a}}$, which are $SU(2)_{\mathcal{R}}$ doublets each containing four real scalars. For simplicity we take diagonal prepotential: $\mathcal{F} = \frac{1}{2}\Phi^i\Phi^i$, without losing generality on the qualitative level. The the effective scalar potential derived from the above Lagrangian is

$$\mathcal{V}_{\text{con}} = 2 \sum_{\hat{a},i,j} n_i^{\hat{a}} n_j^{\hat{a}} \bar{X}^i X^j \mathscr{H}^{\hat{a}\dagger}\mathscr{H}^{\hat{a}} + \frac{1}{4}\sum_i \vec{D}_i \cdot \vec{D}_i \quad \text{with} \quad \vec{D}_i = \sum_{\hat{a}} n_i^{\hat{a}} \mathscr{H}^{\hat{a}\dagger}\vec{\sigma}\mathscr{H}^{\hat{a}}, \tag{4.10}$$

where $\vec{\sigma}$ are Pauli matrices. The flat directions defined by this potential can be separated into the Coulomb branch and the Higgs branch.

The M side of the conifold transition corresponds to the Coulomb branch of the gauge theory, where X^i obtain nonzero vacuum expectation values (VEV's), while $\mathscr{H}^{\hat{a}}$ have vanishing VEV's. This renders the charged hypermultiplets massive through the first term in the potential (4.10). On the other hand the vector multiplets containing X^i are massless, leaving the gauge group $U(1)^S$ unbroken.

On the other hand, the M' side of the conifold transition corresponds to the Higgs branch of the effective field theory, where the vector multiplet scalars X^i have vanishing VEV, while scalars in the charged hypermultiplets $\mathscr{H}^{\hat{a}}$ acquire nonzero VEV's. The latter should be subject to the condition that the D-terms $\vec{D}_i = \sum_{\hat{a}} n_i^{\hat{a}} \mathscr{H}^{\hat{a}\dagger}\vec{\sigma}\mathscr{H}^{\hat{a}}$ vanish. This imposes $3S$ constraints on the $4R$ real variables contained in $\mathscr{H}^{\hat{a}}$, leaving us with $4R - 3S$ flat directions. However all these flat directions do not represent physically inequivalent vacua since points can lie on the same orbit of the gauge group $U(1)^S$. Modding out this gauge redundancy further eliminates S flat directions so we are left with $4(R - S)$, which can be arranged into $R - S$ massless hypermultiplets. Meanwhile the rest S charged hypermultiplets become massive through the $\sum_i \vec{D}_i \cdot \vec{D}_i$ term in Eq.(4.10). The vector multiplets associated to $U(1)^S$ also obtain mass through the $\bar{X}X\mathscr{H}^\dagger\mathscr{H}$ term and they absorb the S massive hypermultiplets, forming S long massive vector multiplets. Therefore the gauge group $U(1)^S$ under which the black hole states are charged is completely "Higgsed away".

Thus going from the Coulomb branch to the Higgs branch, the effective field theory loses S massless vector multiplets and acquires $R - S$ massless hypermultiplets, matching exactly the Hodge number change Eqs (4.4) and (4.5).

4.3 Extremal transition II: uniform singular curves

Topology change

Another type of extremal transition that our work concern is the case where the singular configuration of CY_3 develops a rational curve of ADE-type singularity. This situation arises in the cases where the CY_3 is defined as algebraic variety embedded in some weighted projective space. The ambient space already contains orbifold singularities, and the restriction of the singular locus in the CY_3 is some rational curve. The standard toric geometry procedure [48] to resolve these singularities results in a bunch of $\mathbb{P}^1$'s with ADE-type intersection matrix. We call this nonsingular CY_3 M. On the other hand, the singularities can be deformed into 3-cycles, giving rise to a nonsingular CY_3, to be denoted by M'', of different topology than that obtained from resolution of singularities. We also let $\check{M}$ be the singular configuration connecting M and M''. We denote the mirror manifolds by W, $\check{W}$ and W''.

The cases that we are interested in are those where the intersection matrix of $\mathbb{P}^1$'s in M, arising from the resolution of ambient space singularities, is of type A_{N-1}. Thus we have $N-1$ $\mathbb{P}^1$'s along the singular curve, spanning an $(N-1)$-dimensional subspace of $H_{1,1}(M)$. Let the rational curve be of genus g and we denote the curve by C_g. At the singular locus where the $\mathbb{P}^1$'s in M shrink to zero size leading to $\check{M}$, there are $(g-1)(N^2-N)$ independent non-toric deformations available allowing that the shrinking 2-spheres be deformed into 3-spheres, giving rise to the nonsingular CY_3 M''. Therefore the branch M'' of the extremal transition exists only for $g \geq 2$, and we have the Hodge number change

$$h_{11}(M'') = h_{11}(M) - (N-1), \quad h_{12}(M'') = h_{12}(M) + (g-1)(N^2-N) - (N-1), \quad (g \geq 2). \tag{4.11}$$

Non-perturbative states

We then consider type II strings compactified on such CY_3's, and check the low energy spectrum that they give at the extremal transition locus. We adopt the type IIA picture, and start out from the M-side of the transition. We denote the 2-cycles arising from resolving ambient space singularity by Γ_i $(i = 1, \ldots, N-1)$. We arrange these 2-cycles so that Γ_i and Γ_{i+1} have nonzero intersection. All connected 2-cycles built out of the Γ_i's are of the form $\Gamma_{ij} = \Gamma_i \cup \cdots \cup \Gamma_j$, for $1 \leq i \leq j \leq N-1$, and can be wrapped by BPS D2-branes or anti-D2-branes (obtained by reversing the orientations). The former (latter) are associated to the $(N^2-N)/2$ positive (negative) roots of A_{N-1}, while the perturbative spectrum provides the remaining $N-1$ massless multiplets in the Cartan subalgebra. In the large volume limit of the curve C_g, the model leads to an $\mathcal{N}=2$ theory in six dimensions describing an $SU(N)$ gauge theory [49]. Thus, one can think of the four dimensional case as arising from an additional compactification on the curve C_g, which breaks further half of the supersymmetries. The resulting effective theory is an $\mathcal{N}=2$ $SU(N)$ gauge theory coupled to g hypermultiplets in the adjoint representation [46], and among all the $g(N^2-1)$ hypermultiplets, the $g(N-1)$ Cartan components are perturbative. An analysis of beta function with these field contents show that the effective gauge theory is asymptoticly free when $g = 0$ conformal when $g = 1$ and non asymptoticly free when $g \geq 2$.

Effective gauge theory

We perform field theory analysis to describe qualitatively the consequence of the extremal transition $M \to \check{M} \to M'$ in the effective gauge theory, while analysis based on supergravity will be presented in Chapter 8. With the physics that we expect near the transition locus, that g hypermultiplets in the adjoint of $SU(N)$, we can write down the Lagrangian in $\mathcal{N}_4 = 1$ superspace language

$$
\begin{aligned}
L_{\mathrm{nA}} = \frac{1}{16\pi}\mathrm{Im}\Big[\int d^4\theta\, \Phi^{a\dagger}\exp\big(V^c T^c_{\mathrm{adj}}\big)^{ab}\Phi^b + \frac{1}{2}\int d^2\theta\, W^{a\alpha}W^a_{\ \alpha}\Big] \\
+ \int d^4\theta \Big[H^{a\mathcal{A}\dagger}\exp\big(2\,V^c T^c_{\mathrm{adj}}\big)^{ab}H^{b\mathcal{A}} + \tilde{H}^{a\mathcal{A}}\exp\big(-2V^c T^c_{\mathrm{adj}}\big)^{ab}\tilde{H}^{b\mathcal{A}}\Big] \\
+ i\sqrt{2}\Big(\int d^2\theta\, f^{abc}\,\tilde{H}^{a\mathcal{A}}\Phi^b H^{c\mathcal{A}} + \mathrm{h.c.}\Big), \quad \text{summing over all repeated indices.} \qquad (4.12)
\end{aligned}
$$

Here f^{abc} are the structure constants of $SU(N)$, where $a, b, c = 1, \ldots, N^2 - 1$ are the gauge indices; T^a_{adj} are generating matrices of $SU(N)$ in the adjoint representation whose components are $(T^a_{\mathrm{adj}})^{bc} = -i\,f^{abc}$; (Φ^a, V^a) are the $\mathcal{N}_4 = 2$ vector multiplets of the gauge group $SU(N)$; the hypermultiplets $(H^{a\mathcal{A}}, \tilde{H}^{a\mathcal{A}})$ carry at once the gauge index and the index $\mathcal{A} = 1, \ldots, g$ counting distinct (N^2-1)-plets each in the adjoint of $SU(N)$. We denote the complex scalars in Φ^a by X^a, and the scalars in the hypermultiplet $(H^{a\mathcal{A}}, \tilde{H}^{a\mathcal{A}})$ by $SU(2)_{\mathcal{R}}$ doublet $\mathscr{H}^{a\mathcal{A}}$, each containing four real scalars $h^{a\mathcal{A}u}$ $(u = 1, 2, 3, 4)$. Without loss of generality on the qualitative level, we again take diagonal prepotential: $\mathcal{F} = \Phi^a\Phi^a$. The effective scalar potential obtained from the Lagrangian (4.12) is

$$
\mathcal{V}_{\mathrm{nA}} = [X, \bar{X}]^a[X, \bar{X}]^a + 2[X, h^{\mathcal{A}u}]^a[h^{\mathcal{A}u}, \bar{X}]^a + \frac{1}{4}\vec{D}^a\cdot\vec{D}^a \quad \text{with} \quad \vec{D}^a = -\,i\,f^{abc}\mathscr{H}^{b\mathcal{A}\dagger}\vec{\sigma}\mathscr{H}^{c\mathcal{A}}, \qquad (4.13)
$$

where $X = X^a T^a_{\mathrm{adj}}$, $h^{\mathcal{A}u} = h^{a\mathcal{A}u}T^a_{\mathrm{adj}}$. Like in the previous section, the flat directions can separate into the Coulomb branch and the Higgs branch and we show that they correspond to the type IIA compactification on M and on M'' respectively.

The Coulomb branch is characterized by nonzero VEV's of X^a and $\mathscr{H}^{a\mathcal{A}}$ lying only in Cartan subalgebra of the gauge group. We suppose without loss of generality that the components acquiring nonzero VEV's are X^i and $\mathscr{H}^{i\mathcal{A}}$ $(i = 1, \ldots, N-1)$, with $\{T^i_{\mathrm{adj}}\}$ forming the Cartan subalgebra of $SU(N)$. The VEV's of X^i supply $N-1$ complex directions in the vector multiplet moduli space $\mathcal{M}_{\mathrm{V}}$ and those of $\mathscr{H}^{i\mathcal{A}}$ supply $g(N-1)$ quaternionic directions in the hypermultiplet moduli space

$\mathcal{M}_{\mathrm{H}}$. The rest of the N^2-N vector multiplets and $g(N^2-N)$ hypermultiplets that are non Cartan obtain mass. This is just the situation arising from the compactification on M, where the massive non Cartan components are from D2-brane wrapping shrinking 2-spheres. At generic values of X^i and $\mathscr{H}^{i\mathcal{A}}$, all the non Cartan vector multiplets become massive, and the gauge group is broken to $U(1)^{N-1}$. It is possible to assign VEV's which give mass only to part of the non Cartan components, so that the gauge group is a rank $N-1$ subgroup of $SU(N)$. Also there is a subtlety about the massive field content: when we switch on VEV's only for X^i, the N^2-N non Cartan vector multiplets acquiring mass stay short, while if we switch on VEV's also for $\mathscr{H}^{i\mathcal{A}}$, each of the non Cartan vector multiplet absorbs a non Cartan hypermultiplet and become long vector multiplets.

Then we examine the Higgs branch, where all $SU(N)$-vector multiplet scalars have VEV's fixed at 0, while all $SU(N)$-hypermultiplet scalars are allowed to move freely in the flat directions defined by $\vec{D}^a = 0$. This imposes $3(N^2-1)$ constraints on the $4g(N^2-1)$ real degrees of freedom included in $\mathscr{H}^{a\mathcal{A}}$. Among the $(4g-3)(N^2-1)$ real flat directions that rest, not all of them parameterize physically inequivalent vacua because they accommodate the orbit of gauge group $SU(N)$. Fixing this gauge freedom wipes out another N^2-1 real flat directions. Therefore we have all physically inequivalent vacua parameterized by $4(g-1)(N^2-1)$ real flat directions, which can be arranged into $(g-1)(N^2-1)$ massless hypermultiplets. This counting shows that the Higgs branch exists only for $g \geq 2$. On the other hand, all $SU(N)$-vector multiplets obtain masses through the second term in the potential (4.13) and the rest N^2-1 charged hypermultiplets obtain masses through the $\vec{D}^a \cdot \vec{D}^a$ term. Further, these massive fields are combined and form N^2-1 long vector multiplets. Thus the whole gauge group $SU(N)$ is "Higgsed away".

Therefore when moving into the Higgs branch from the Coulomb branch, we loose the $N-1$ massless vector multiplets and the $g(N-1)$ massless hypermultiplets in a Cartan subalgebra of $SU(N)$, but meanwhile we gain $(g-1)(N^2-1)$ massless hypermultiplets. Thus the Kähler moduli are reduced by $N-1$ while the complex structure moduli are increased by $(g-1)(N^2-1)-g(N-1) = (g-1)(N^2-N)-(N-1)$, corresponding exactly to the Hodge number change Eq.(4.11). Thus the compactification on M'' gives rise to the Higgs branch of the effective gauge theory.

Chapter 5

Ideal string gas at finite temperature

For cosmological application of string theory, we need a thermodynamical description of string gas in a closed system. The goal of this chapter is to clarify this issue. We adopt the canonical ensemble prescription, where the derivation of all thermodynamical quantities is based on the partition function

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H}. \tag{5.1}$$

Here $\beta = 1/T$ is the inverse temperature and H is the Hamiltonian of the string gas. Once the partition function is obtained, the thermodynamics of string gas follows the standard formalism. We can derive from the partition function the Helmholtz free energy F and its density

$$F = -T \ln \mathcal{Z} = -\frac{Z}{\beta}, \quad \mathcal{F} = \frac{F}{V} = -\frac{Z}{\beta V}, \tag{5.2}$$

with $Z = \ln\mathcal{Z}$ and V the space volume. We point out here that the free energy density will play a key role in cosmological application. Based on $\mathcal{F}$ we have the pressure P and the energy density of the gas, given by

$$P = -\frac{\partial F}{\partial V} = -V \frac{\partial \mathcal{F}}{\partial V} - \mathcal{F}, \quad \rho = -\frac{\partial}{\partial \beta} \ln\mathcal{Z} = T \frac{\partial P}{\partial T} - P. \tag{5.3}$$

These are just the quantities appearing in the Friedman equations sourcing the cosmological evolution.

A direct evaluation of the partition function in second quantization formalism should appeal to string field theory. However here we would rather adopt a first quantization approach of which the technique is better established. We will show, through point particle analogue, that this first quantization alternative is feasible at weak coupling regime, which is because $\ln\mathcal{Z}$ for an ideal gas is just a one-loop amplitude computed against a Euclidean background.

5.1 Lesson from point particle

Partition function as one-loop amplitude

We recall the standard result of $\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H}$, the canonical partition function for an ideal gas of particle of one degree of freedom:

$$\ln \mathcal{Z} = (-)^{F+1} \frac{V_{D-1}}{(2\pi)^{D-1}} \int d\vec{p} \left\{ \frac{1}{2}\beta\omega_p + \ln\left[1 - (-1)^F e^{-\beta\omega_p}\right] \right\}, \tag{5.4}$$

where V_{D-1} is the space volume, $d\vec{p} = d^{D-1}p$, $\omega_p = \sqrt{\vec{p}^2 + M^2}$ with M the particle mass, and F takes value 0 for bosonic degree of freedom, 1 for fermionic. The explicit computation is presented in Appendix A.1. The result is easily obtained by path integral against a Euclidean background with the Euclidean time circle of perimeter $\beta = 2\pi R_0$, where periodic (anti-periodic) boundary condition is assigned to the bosonic (fermionic) field. This result has the interpretation of one-loop vacuum amplitude, which can be seen very explicitly when we expand the logarithmic in power series of $e^{-\beta\omega_p}$:

$$\ln \mathcal{Z} = -(-)^F \frac{V_{D-1}}{(2\pi)^{D-1}} \int d\vec{p}\, \frac{\beta\omega_p}{2} + \frac{V_{D-1}}{(2\pi)^{D-1}} \sum_{n=1}^{\infty} \frac{(-1)^{nF+F}}{n} \int d\vec{p}\; e^{-n\beta\,\omega_p} \tag{5.5}$$

The physical significance of the summation on the right hand side can be better elucidated by noticing, for the n-th term, that

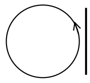
$$\frac{V_{D-1}}{(2\pi)^{D-1}} \frac{1}{n} \int d\vec{p}\; e^{-n\beta\,\omega_p} = \frac{1}{n} \int d\vec{p}\, \langle \vec{p}\,| {:} e^{-n\beta\,H} {:} |\vec{p}\,\rangle = \frac{1}{n} \int d\vec{x}\, \langle \vec{x}\,| {:} e^{-n\beta\,H} {:} |\vec{x}\,\rangle, \tag{5.6}$$

where $|\vec{p}\,\rangle$ and $|\vec{x}\,\rangle$ are single particle states in the momentum and position representation, appropriately normalized. This is the amplitude that a particle launched at any spatial point, circulating the Euclidean time circle n-times ($n = 1, 2, \dots$) before finally returning to the initial point. The division by n is because in $\langle \vec{x}\,| {:} e^{-n\beta\,H} {:} |\vec{x}\,\rangle$ the cases that a particle launched in the 1st, the 2nd,..., until the n-th β-interval of Euclidean time are redundantly counted. The first term on the right hand side of Eq.(5.5) can be considered, referring to Eq.(A.16), as the $n \to 0$ limit of Eq.(5.6), up to a physically irrelevant infinity. Therefore it is from the particle loop which does not wind around the Euclidean time circle, i.e. the vacuum bubble. Therefore Eq.(5.5) is the total one-loop amplitude of a single particle in the ideal gas, with the subtlety for the fermionic case that the winding number n is summed up with interchanging signs, which is the consequence of spin-statistics.

First quantization: an alternative approach

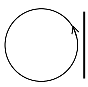
With the one-loop amplitude interpretation of the logarithmic of the partition function, we are now attempted to write down

$$\ln \mathcal{Z} = \quad (5.7)$$



$$\Big|_{\beta} \,,$$

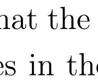
where the diagram on the right hand side is understood as obtained by first quantization formalism with boundary conditions satisfying spin-statistics. Indeed a direct first quantization evaluation shows that (see detailed computation in appendix A.2)



$$\Big|_{\beta} = \frac{\beta V_{D-1}}{2(2\pi)^D} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{d\ell}{\ell^{D/2+1}} (-1)^{F(n+1)} \exp\Big(-\frac{\pi R_0^2}{\ell} n^2 - \pi\,\ell\, M^2\Big). \quad (5.8)$$

This is nothing but the Schwinger parameter representation of the standard formula (5.4), where ℓ is just the Schwinger parameter, proportional to the size of the particle loop. Here the result Eq.(5.8) can be obtained more easily by second quantization path integral, as is shown in Eqs(A.8) and (A.10). Indeed it is shown in the same appendix that the n-th term in the summation in Eq.(5.8) is identical to the n-th term in the summation of second quantization result Eq.(5.5). Therefore the first quantization and second quantization evaluations of $\ln\mathcal{Z}$ match, confirming the validity of the schema Eq.(5.7). The significance of this statement is that in case where a second quantization evaluation of partition function is not possible or not convenient, there can be a first quantization alternative, where one evaluates the one-loop amplitude against a thermal background. We will see in the next section that this is just the case for ideal string gas.

However before turning to the string gas, we still have lesson to draw from point particle. We need to consider an ideal gas consisting of several species of particles. In such case the total one-loop amplitude should be the sum of the amplitude of each degree of freedom. This is due to the additivity of $\ln\mathcal{Z}$, the consequence of the fact that the whole Hilbert space is a tensor product of the Hilbert spaces of each degree of freedom. Therefore if we let each degree of freedom in the ideal gas is labeled by s, with mass spectrum $\{M_s\}$ and fermionic numbers $\{F_s\}$, the total one-loop amplitude reads

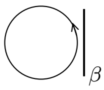


$$\Big|_{\beta} = \frac{\beta V_{D-1}}{2(2\pi)^D} \sum_s \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{d\ell}{\ell^{D/2+1}} (-1)^{F_s n + F_s} \exp\Big(-\frac{\pi R_0^2}{\ell} n^2 - \pi\,\ell\, M_s^2\Big). \quad (5.9)$$

This shows that the one-loop amplitude for an ideal particle gas cares only about the spin and the mass of states in the spectrum. That is, for any ideal gas of point particles, once the spectrum of the system is known, we can obtain its thermal one-loop amplitude by plugging in the masses and fermion numbers into the above formula. However we will see in the next section that for an ideal string gas, its thermalization follows more subtle steps inspired by the formalism Eq.(5.9). To handle the subtleties in string theory, we need to go somewhat further in the interpretation of Eq.(5.9).

A closer look at thermalization

We first rewrite Eq.(5.9) in a form displaying more details:



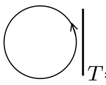
$$\Big|_{\beta} = \mathcal{A}_0 + \mathcal{A}_1, \quad \text{where} \tag{5.10}$$

$$\mathcal{A}_0 = \frac{\beta V_{D-1}}{2(2\pi)^D} \int_0^\infty \frac{d\ell}{\ell^{D/2+1}} \Bigg[\sum_b \exp\Big(- \pi\, \ell\, M_b^2 \Big) - \sum_f \exp\Big(- \pi\, \ell\, M_f^2 \Big) \Bigg], \tag{5.11}$$

$$\mathcal{A}_1 = \frac{\beta V_{D-1}}{2(2\pi)^D} \int_0^\infty \frac{d\ell}{\ell^{D/2+1}} \sum_{n\neq 0} \Bigg[\sum_b \exp\Big(- \frac{\pi R_0^2}{\ell} n^2 - \pi\, \ell\, M_b^2 \Big) - (-1)^n \sum_f \exp\Big(- \frac{\pi R_0^2}{\ell} n^2 - \pi\, \ell\, M_f^2 \Big) \Bigg]. \tag{5.12}$$

By distinguishing $\mathcal{A}_1$ from $\mathcal{A}_0$ we separated the thermal part from the vacuum bubble contribution, and in each piece we further distinguished the bosonic contribution from the fermonic contribution. In particular, we let the index s in Eq.(5.9) be split into bosonic indices and fermonic ones: $\{s\} = \{b\} \cup \{f\}$, so that $F_b \equiv 0$ and $F_f \equiv 1$. Eq.(5.11) shows that the vacuum bubble contribution generically has UV ($\ell \to 0$) divergence unless some cancelation happens between bosons and fermions, for example in case of supersymmetry, or unless the spectrum contains an infinity of states, the infinite sum over which regularizes the divergence. This above expression shows that the thermal one-loop amplitude is obtained by pending a thermal piece $\mathcal{A}_1$ to the non thermal piece $\mathcal{A}_0$ where the information needed to construct $\mathcal{A}_1$, the masses and the fermion numbers, can be simply read off from $\mathcal{A}_0$. Certainly we also have to take into account the change in spacetime volume: $V_D \to \beta V_{D-1}$. This method is perfectly designed for free string theory, where we know the technique to compute the zero temperature one-loop amplitude as is shown in Chapter 2.

Another instructive point of view provided by Eq.(5.9), is based on instanton interpretation of thermal effect. To see this we compare Eq.(5.9) the zero temperature case. By switching off the temperature letting $\beta \to \infty$ in Eq.(5.9), the terms with $n \neq 0$ are exponentially suppressed, so that we get



$$\Big|_{T=0} = \frac{V_D}{2(2\pi)^D} \sum_s \int_0^\infty \frac{d\ell}{\ell^{D/2+1}} (-1)^{F_s} \exp\Big(- \pi\, \ell\, M_s^2 \Big), \tag{5.13}$$

which is just the pure vacuum bubble contribution Eq.(5.11). This conforms to the fact that a particle feels temperature only when it circulates the Euclidean time circle, while the vacuum bubbles tell nothing about the temperature. Comparing to the $T \neq 0$ case in Eq.(5.9), we find that operationally the temperature is turned on by the inserting into Eq.(5.13) a weighted worldline instanton sum

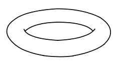
$$\sum_n (-1)^{F_s n} \exp\Big(-\frac{\pi R_0^2}{\ell} n^2\Big) \tag{5.14}$$

and at the same time taking into account the change in spacetime volume: $V_D \to \beta V_{D-1}$. This insertion corresponds to the compactification on the Euclidean time circle, with anti-periodic boundary condition assigned to fermions. In string theory application, the worldline instanton sum will be replaced by worldsheet instanton sum.

5.2 Ideal gas of closed superstrings

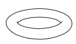
With all the preliminary analysis done for point particle, we are at the point of addressing the thermodynamics of ideal string gas. The main focus is the computation of the partition function Eq.(5.1). The discussion in the last section, which leads to the schema Eq.(5.7), shows that the computation can be achieved by first quantization formalism. The starting point of technical computation is to coin up a string version of Eq.(5.7).

In this section we do this for a gas of closed strings and the open string gas will be discussed in the next section. In fact cases of open strings are relatively easier since open strings behave more like point particles, while closed strings show more stringy subtleties, in that closed string can wrap the Euclidean time circle, what point particles and open strings cannot do, and also the resulting one-loop amplitude has to be invariant under modular group $SL(2,\mathbb{Z})$. To begin with, we write down the closed string version of Eq.(5.7)

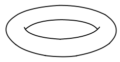
$\ln \mathcal{Z} =$  $\Big|_\beta$.   (5.15)

In the rest of this section we discuss the computation of this one-loop amplitude.

Intuitive thermodynamical argument

To obtain the amplitude  $|_\beta$, a naive idea is taking Eq.(5.9), and plugging in the mass spectrum of the closed string model. However this approach is not correct beause the result is not modular invariant, moreover we do not really recover the zero temperature vacuum amplitude at $T \to 0$ limit, and the result suffers from UV divergence. It means that this prescription fails to give proper credit to the good UV behavior of string theory. An obvious example is when spacetime supersymmetry is spontaneously broken, the vacuum amplitude is nontrivial and finite, but when applying Eq.(5.9) to this case, we observe UV divergence in $n = 0$ terms.

We should proceed more delicately to avoid the pathology in the naive recipe. The improved version is to first take Eq.(5.10), then plug the string mass spectrum only into $\mathcal{A}_1$, while let $\mathcal{A}_0$ be simply the one-loop vacuum amplitude at $T = 0$ whose computation is presented especially in Chapter 2, and finally let the thermal one-loop amplitude be the sum of the two. Thus obviously the zero temperature limit yields the correct result. However we do not know whether $\mathcal{A}_0 + \mathcal{A}_1$ is modular invariant although $\mathcal{A}_0$ is. Indeed it is the case, and we will show this later. Now we describe the technical steps in more detail. To distinguish from point particle case, we denote the string theory counterpart of $\mathcal{A}_0$ and $\mathcal{A}_1$ by Z_0 and Z_1. Mimicking Eq.(5.10), we write down



$$\Big|_{\beta} := Z(T) = Z_0 + Z_1 \,. \tag{5.16}$$

For the zero temperature part Z_0, we can assume its generic form

$$Z_0 = \frac{\beta V_{D-1}}{2(2\pi)^D} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} \left[\sum_{B,\bar{B}} \mathcal{P}_{B\bar{B}}\, q^B \bar{q}^{\bar{B}} - \sum_{F,\bar{F}} \mathcal{Q}_{F\bar{F}}\, q^F \bar{q}^{\bar{F}} \right] \tag{5.17}$$

which is always the case for all closed string models that we have or have not encountered. In the bracket, the two terms represent the spacetime boson contribution and the spacetime fermion contribution respectively. The powers $\{B, \bar{B}\}$ and $\{F, \bar{F}\}$ run through discrete values that are model dependent. The multiplicities of any specific power of q and $\bar{q}$ are $\{\mathcal{P}_{B\bar{B}}\}$ and $\{\mathcal{Q}_{F\bar{F}}\}$ for bosonic and fermonic contribution respectively. All multiplicities are positive integers. In the case where spacetime supersymmetry is unbroken, the permissible values of powers are the same between the bosonic and the sermonic sector: $\{B, \bar{B}\} = \{F, \bar{F}\}$, and the corresponding multiplicities are also identical: $\{\mathcal{P}_{B\bar{B}}\} = \{\mathcal{Q}_{F\bar{F}}\}$. As a result Eq.(5.17) vanishes. To compute Z_1, we need to read off the mass spectrum from Eq.(5.17) and insert into Eq.(5.12). The physical states should be level matched: $B = \bar{B}$ and $F = \bar{F}$. Thus the mass spectrum communicated by Eq.(5.12) is that at the level B (F), the mass is $2\sqrt{B}$ ($2\sqrt{F}$), with degeneracy $\mathcal{P}_{BB}$ ($\mathcal{Q}_{FF}$). Thus inserting these data into Eq.(5.12), we get the thermal piece

$$Z_1 = \frac{\beta V_{D-1}}{2(2\pi)^D} \int_0^\infty \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \sum_{\tilde{m}_0 \neq 0} \left[\sum_B \mathcal{P}_{BB}\, e^{-\frac{\pi R_0^2}{\tau_2} - \pi\tau_2(4B)} - (-1)^{\tilde{m}_0} \sum_F \mathcal{Q}_{FF}\, e^{-\frac{\pi R_0^2}{\tau_2} - \pi\tau_2(4F)} \right]. \tag{5.18}$$

The steps presented above are more for giving general principles that the thermal one-loop amplitudes should respect, than for providing technical algorithm for explicit computation. Indeed, in practice when dealing with specific models, these steps are not convenient to operate. It is because the computation of Z_1 as in Eq.(5.18) requires a full description of the mass spectrum, i.e. specifying each mass level and its degeneracy, which is a very complicated thing to achieve.

In explicit computation we will go through another way round, which is inspired by the discussion in the end of the last section giving rise to Eqs (5.13) and (5.14). This amounts to first taking the $T = 0$ amplitude of the closed string that we want to thermalize, and then, mimicking the insertion of Eq.(5.14) in the point particle case, inserting a worldsheet instanton sum into the $T = 0$ amplitude. The inserted instanton sum should correspond to the compactification on the Euclidean time on a circle, with a suitable phase weighting the instanton sum in order to respect spin-statistics. This construction is more stringy than thermodynamical, but in the end we will show that one recovers precisely, from the stringy results, Eqs (5.17) and (5.18) from intuitive thermodynamical argument.

Thermalization of ideal heterotic and type II string gas

We consider thermalization of ideal gas of the heterotic string and type II strings. For the heterotic string, the one-loop amplitude at zero temperature is given by Eq.(2.55). With the compactification on the Euclidean time circle, the instanton sum to be inserted is as the right hand side of Eq.(3.3), with the subscripts 9 replaced by 0. Taking also into account spin-statistics, there should be a phase insertion in the instanton, and the phase is further constrained by $SL(2,\mathbb{Z})$ modular invariance of one-loop amplitudes. We give directly the weighted instanton sum, which turns out to be

$$\sum_{\tilde{m}_0,n_0}(-1)^{a\tilde{m}_0+bn_0+\tilde{m}_0n_0}\exp\left(-\frac{\pi R_0^2}{\tau_2}\left|\tilde{m}_0-\tau n_0\right|^2\right), \tag{5.19}$$

Note that a and b are the indices summed over in Eq.(2.55), which indicate the spin structures along the two independent cycles of the worldsheet which is of the topology of torus. Therefore they are the string theory counterpart of the fermion number F_s in Eq.(5.14). The cycle of spin structure a is the transversal direction of the string, which wraps n_0 times the Euclidean time circle, while that of spin structure b is in the propagating direction of the string, wraping $\tilde{m}_0$ times. This motivates the expression $a\,\tilde{m}_0 + b\,n_0$ in the phase in Eq.(5.19), in analogy to F_sn in Eq.(5.14). It is natural that the spin structure indices a and b appear in the phase on equal footing, since a and b are transformed into each other through the modular transformation $\tau \to -1/\tau$. The additional $\tilde{m}_0n_0$ term in the phase is included just for sake of modular invariance. A more rigorous way to obtain the phase Eq.(5.19) is to postulate the phase $(-1)^{a\tilde{m}_0}$ for $n_0 = 0$, and then use modular invariance to find out the full expression for all values of n_0. This is done in [50] for the heterotic string, although the result is expressed in a different form. Therefore the thermal partition function is

$$Z_{\rm h}(T) = \frac{\beta V_{D-1}}{2(2\pi)^D}\int_{\mathcal F}\frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}}\sum_{a,b}(-1)^{a+b+ab}\frac{\theta[^a_b]^4}{2\eta^4}\left[\sum_{\tilde m_0,n_0}(-1)^{a\tilde m_0+bn_0+\tilde m_0 n_0}\,e^{-\frac{\pi R_0^2}{\tau_2}|\tilde m_0-\tau n_0|^2}\right]\frac{\Gamma_{(d,16+d)}}{\eta^8\bar\eta^{24}}. \quad (5.20)$$

In type II strings, the spacetime fermion number is $a+\bar a$ as is clarified below Eq.(2.38). Thus by the same reasoning as for the heterotic string, the weighted instanton sum to be inserted into Eq.(2.39) is

$$\sum_{\tilde m_0,n_0}(-1)^{(a+\bar a)\tilde m_0+(b+\bar b)n_0}\,\exp\left(-\frac{\pi R_0^2}{\tau_2}\big|\tilde m_0-\tau n_0\big|^2\right), \quad (5.21)$$

so that the thermal one-loop amplitude is

$$\begin{aligned} Z_{\rm II}(T) = \frac{\beta V_{D-1}}{2(2\pi)^D}\int_{\mathcal F}\frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}}&\sum_{a,b}(-1)^{a+b+ab}\frac{\theta[^a_b]^4}{2\eta^4}\sum_{\bar a,\bar b}(-1)^{\bar a+\bar b+\mu\bar a\bar b}\frac{\bar\theta[^{\bar a}_{\bar b}]^4}{2\bar\eta^4}\\ &\times\left[\sum_{\tilde m_0,n_0}(-1)^{(a+\bar a)\tilde m_0+(b+\bar b)n_0}\,e^{-\frac{\pi R_0^2}{\tau_2}|\tilde m_0-\tau n_0|^2}\right]\frac{\Gamma_{(d,d)}}{\eta^8\bar\eta^8}\,. \end{aligned} \quad (5.22)$$

We have two remarks to add. First, the string models considered above can possibly be compactified on some internal torus before switching on temperature. However actually the scheme of thermalization can by all means be applied to models compactified on any internal space as long as the noncompact spacetime is flat, since obviously, the insertions of Eqs (5.19) and (5.21) do not care about the internal space. The models giving rise to flat spacetime solutions are just the no-scale type models [8]. Second, by construction the above finite temperature results Eqs (5.20) and (5.22) are modular invariant. In fact referring to the discussion at the end of Sec.3.3, we see that switching on temperature amounts exactly to the implementation of a Scherk-Schwarz reduction $S^1(2R_0)/Z_2$. Here the orbifold action is as in Eq.(3.12), with δ the order two translation along the Euclidean time circle of radius $2R_0$, and Q the spacetime fermion number.

Back to intuition on thermodynamics

We still need to show that the results Eqs (5.20) and (5.22) obtained in a stringy way makes sense thermodynamically. That is, we verify that they conform to the initially advertised forms Eqs (5.17) and (5.18). We observe that Eqs (5.20) and (5.22) take the form

$$Z(T) = \int_{\mathcal F}\frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}}\sum_{\tilde m_0,n_0}f_{(\tilde m_0,n_0)}(\tau,\bar\tau,\dots), \quad (5.23)$$

where the integrand $\sum f_{(\tilde{m}_0,n_0)}(\tau,\bar{\tau},\dots)$ is modular invariant. In fact, let $\tau' = (a\tau+b)/(c\tau+d)$ with $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, then $f_{(\tilde{m}_0,n_0)}(\tau',\bar{\tau}',\dots) = f_{(\tilde{m}_0,n_0)M^{-1}}(\tau,\bar{\tau},\dots)$. In such cases one can apply the unfolding technique [51] to decompose the instanton sum in the following way

$$Z(T) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} f_{(0,0)}(\tau,\bar{\tau},\dots) + \int_{\sqcup} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} \sum_{\tilde{m}_0\neq 0} f_{(\tilde{m}_0,0)}(\tau,\bar{\tau},\dots), \tag{5.24}$$

where in the second term the integral domain is the strip $-\frac{1}{2} < \tau_1 < \frac{1}{2}$ and $\tau_2 > 0$. Clearly the first term is just the zero temperature vacuum amplitude so that when expanding the integrand $f_{(0,0)}(\tau,\bar{\tau},\dots)$ in powers of q and $\bar{q}$ we find exactly Eq.(5.17):

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} f_{(0,0)}(\tau,\bar{\tau},\dots) = \frac{\beta V_{D-1}}{2(2\pi)^D} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} \left[\sum_{B,\bar{B}} \mathcal{P}_{B\bar{B}}\, q^B \bar{q}^{\bar{B}} - \sum_{F,\bar{F}} \mathcal{Q}_{F\bar{F}}\, q^F \bar{q}^{\bar{F}} \right] \tag{5.25}$$

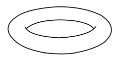
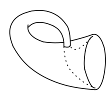
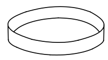
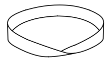
In the second term in Eq.(5.24), only the sum over winding number $\tilde{m}_0$ is concerned. Therefore the effect is to insert into the above expression the instanton sum $\sum'_{\tilde{m}_0} \exp\left(-\frac{\pi R_0^2}{\tau 2}\tilde{m}_0^2\right)$, and assigning the phase $(-1)^{\tilde{m}_0}$ to the fermonic sector

$$\begin{aligned}
&\int_{\sqcup} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} \sum_{\tilde{m}_0\neq 0} f_{(\tilde{m}_0,0)}(\tau,\bar{\tau},\dots) \\
&= \frac{\beta V_{D-1}}{2(2\pi)^D} \int_{\sqcup} \frac{d^2\tau}{\tau_2^{1+\frac{D}{2}}} \sum_{\tilde{m}_0}{}' \left[\sum_{B,\bar{B}} \mathcal{P}_{B\bar{B}}\, q^B \bar{q}^{\bar{B}} - (-1)^{\tilde{m}_0} \sum_{F,\bar{F}} \mathcal{Q}_{F\bar{F}}\, q^F \bar{q}^{\bar{F}} \right] e^{-\frac{\pi R_0^2}{\tau_2}\tilde{m}_0^2} \\
&= \frac{\beta V_{D-1}}{2(2\pi)^D} \int_0^\infty \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \sum_{\tilde{m}_0}{}' \left[\sum_{B} \mathcal{P}_{BB}\, (q\bar{q})^B - (-1)^{\tilde{m}_0} \sum_{F} \mathcal{Q}_{FF}\, (q\bar{q})^F \right] e^{-\frac{\pi R_0^2}{\tau_2}\tilde{m}_0^2} \\
&= \frac{\beta V_{D-1}}{2(2\pi)^D} \int_0^\infty \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \sum_{\tilde{m}_0}{}' \left[\sum_{B} \mathcal{P}_{BB}\, e^{-\frac{\pi R_0^2}{\tau_2} - \pi\tau_2(4B)} - (-1)^{\tilde{m}_0} \sum_{F} \mathcal{Q}_{FF}\, e^{-\frac{\pi R_0^2}{\tau_2} - \pi\tau_2(4F)} \right].
\end{aligned} \tag{5.26}$$

Here when going from the second to the the third line, we have just performed the integration over τ_1, which imposes the level matching condition. This is possible because the integration domain is unfolded on the strip. Finally we obtain precisely Eq.(5.18), so that the splitting Eq.(5.24) is just the splitting between Z_0 and Z_1.

5.3 Ideal gas of type I string

In Sec.2.4 we have mentioned that the type I string one-loop amplitude at zero temperature can be decomposed into a sum of four amplitudes with different worldsheet topologies. We can implement finite temperature to these four amplitudes separately, and the canonical partition function is obtained through

$\ln \mathcal{Z} := Z_{\mathrm{I}}(T) = \Bigg[$  $+$  $+$  $+$  $\Bigg]_{\beta}$, (5.27)

which again is motivated by Eq.(5.7) in the point particle case.

In the closed string sector, we have the torus amplitude and the Klein bottle amplitude. The former is nothing but the type IIB amplitude, whose thermalization is obtained in the last section:

$$\mathcal{T}(T) = \frac{1}{2} Z_{\mathrm{II}}(T), \tag{5.28}$$

with $Z_{\mathrm{II}}(T)$ given in Eq.(5.22). To thermalize the Klein bottle amplitude, we recall the remark made below Eq.(2.49), that only NS-NS sector and RR sector, i.e. spacetime bosons, contribute to the amplitude. Therefore what we are left to do is just inserting the instanton sum arising from compactification on $S^1(R_0)$ with only bosonic statistics, so we have

$$\mathcal{K}(T) = \frac{\beta V_{D-1}}{2(2\pi)^D} \frac{1}{2} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \sum_{\tilde{m}_0} e^{-\frac{\pi R_0^2}{\tau_2} \tilde{m}_0^2} \sum_{a,b} (-1)^{a+b+ab} \frac{\theta[^a_b]^4}{2\eta^4} \frac{\Gamma_d}{\eta^8}. \tag{5.29}$$

In the open string sector, we observe from the zero temperature one-loop amplitudes (2.50) and (2.51), that open strings behave just like point particles, so that the point particle scheme can be simply transplanted to this case. The spacetime fermion number being a in Eqs (2.50) and (2.51), we insert the instanton sum as Eq.(5.14) with $n \to \tilde{m}_0$ and $F_s \to a$. Thus,

$$\mathcal{A}(T) = \frac{\beta V_{D-1}}{2(2\pi)^D} \frac{N^2}{2} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \sum_{a,b} \Big[\sum_{\tilde{m}_0} (-1)^{a\tilde{m}_0} e^{-\frac{\pi R_0^2}{\tau_2} \tilde{m}_0^2} \Big] (-1)^{a+b+ab} \frac{\theta[^a_b]^4}{2\eta^4} \frac{\Gamma_d}{\eta^8}, \tag{5.30}$$

$$\mathcal{M}(T) = \frac{\beta V_{D-1}}{2(2\pi)^D} \frac{\zeta N}{2} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \sum_{a,b} \Big[\sum_{\tilde{m}_0} (-1)^{a\tilde{m}_0} e^{-\frac{\pi R_0^2}{\tau_2} \tilde{m}_0^2} \Big] (-1)^{a+b+ab} \frac{\hat{\theta}[^a_b]^4}{2\hat{\eta}^4} \frac{\Gamma_d}{\hat{\eta}^8}. \tag{5.31}$$

The results Eqs (5.28)–(5.31) can also be considered as the implementation of Scherk-Schwarz reduction $S^1(2R_0)/Z_2$ just as in the closed string case. The total one-loop amplitude at finite temperature is the sum of Eqs (5.28)–(5.31)

$$Z_{\mathrm{I}}(T) = \mathcal{T}(T) + \mathcal{K}(T) + \mathcal{A}(T) + \mathcal{M}(T). \tag{5.32}$$

It is straightforward to see that this result for type I string conforms to the intuition from thermodynamics that is, it can be separated into a thermal piece and a non-thermal piece, where the former is the $T = 0$ amplitude, and the latter takes the form Eq.(5.18). The case of $\mathcal{T}(T)$ is already explained by the end of last section in Eq.(5.26). In the rest amplitudes $\mathcal{K}(T)$, $\mathcal{A}(T)$ and $\mathcal{M}(T)$, we identify the non-thermal part which is the $\tilde{m}_0 = 0$ part, and for the rest with $\tilde{m}_0 \neq 0$, we expand them in power series in q, and result takes exactly the form of Eq.(5.18).

5.4 Hagedorn singularity

A remarkable feature of the thermal one-loop amplitudes computed above is that the sum over the mass spectrum diverges when temperature goes high enough. It is due to the characteristic of string mass spectrum, that the degeneracy grows exponentially with mass: degen.$(M) \sim e^{\text{const.}\times M}$. On the other hand the contribution to the one-loop amplitude from a heavy state of mass M, satisfying $M \gg T$, is always dressed by a Boltzmann factor $e^{-M/T}$. When computing the one-loop amplitude, at each mass level we sum up $e^{\text{const.}\times M}$ times the exponentially suppressed Boltzmann factor $e^{-M/T}$, and then we sum over all mass levels until $M \to \infty$. Thus when the temperature is above a critical value, the number of states $e^{\text{const.}\times M}$ wins over the Boltzmann factor $e^{-M/T}$ so that the sum over M diverges. The singularity in the thermal partition sum arising from the above mechanism is named after Hagedorn, who first unraveled this phenomenon in QCD [52]. The critical temperature is referred to as Hagedorn temperature, which we will denote by $T_{\text{H}} = 1/\beta_{\text{H}}$.

Hagedorn singularity as UV effect

Here we describe in more detail the qualitative discussion made above, and we first examine the closed strings. Recall that we have separated the thermal one-loop amplitude $Z(T)$ into a zero temperature part Z_0 and a thermal part Z_1, as in Eq.(5.16). Obviously the divergence comes from the thermal part. For convenience in the following discussion, we change the notation in Eqs (5.18) and (5.26), rewriting the degeneracies as $\mathcal{P}_{BB} \to \mathcal{P}(M_{\text{b}})$ and $\mathcal{Q}_{FF} \to \mathcal{Q}(M_{\text{f}})$ for the bosonic sector and the fermonic sector respectively where $M_{\text{b,f}}$ are mass of states. This is legitimate because as is stated above Eq.(5.18), B and F being the powers of $q\bar{q}$ are related unambiguiously to the mass by $M_{\text{b}} = 2\sqrt{B}$ and $M_{\text{f}} = 2\sqrt{F}$. Thus we rewrite Eq.(5.18) with the new notation and perform the integration over τ_2 using Eq.(A.53), and this yields

$$\begin{aligned} Z_1 =& \frac{\beta V_{D-1}}{2(2\pi)^D}\int_0^\infty \frac{d\tau_2}{\tau_2^{1+\frac{D}{2}}} \sum_{\tilde{m}_0}{}' \left[\sum_{M_{\text{b}}} \mathcal{P}(M_{\text{b}})\, e^{-\frac{\pi R_0^2}{\tau_2}\tilde{m}_0^2 - \pi\tau_2 M_{\text{b}}^2} - (-1)^{\tilde{m}_0} \sum_{M_{\text{f}}} \mathcal{Q}(M_{\text{f}})\, e^{-\frac{\pi R_0^2}{\tau_2}\tilde{m}_0^2 - \pi\tau_2 M_{\text{f}}^2} \right] \\ =& \beta^{1-D} V_{D-1} \left[\sum_{M_{\text{b}}} \mathcal{P}(M_{\text{b}})\, W_{\text{b}}(\beta M_{\text{b}}) + \sum_{M_{\text{f}}} \mathcal{Q}(M_{\text{f}})\, W_{\text{f}}(\beta M_{\text{f}}) \right], \quad \text{where} \end{aligned} \tag{5.33}$$

$$\begin{aligned} W_{\text{b}}(\beta M_{\text{b}}) &= 2 \sum_{\tilde{m}_0=1}^{\infty} \left(\frac{\beta M_{\text{b}}}{2\pi \tilde{m}_0} \right)^{\frac{D}{2}} K_{\frac{D}{2}}\big(\tilde{m}_0\, \beta M_{\text{b}}\big), \\ W_{\text{f}}(\beta M_{\text{f}}) &= 2 \sum_{\tilde{m}_0=1}^{\infty} (-1)^{\tilde{m}_0+1} \left(\frac{\beta M_{\text{f}}}{2\pi \tilde{m}_0} \right)^{\frac{D}{2}} K_{\frac{D}{2}}\big(\tilde{m}_0\, \beta M_{\text{f}}\big), \end{aligned} \tag{5.34}$$

where $K_\nu(\,\cdot\,)$ is the modified Bessel function of the second kind. In fact, We report the asymptotic behavior of the quantities as mass tends to infinity:

$$\mathcal{P}(M), \mathcal{Q}(M) \sim \text{const.} \times M^{-D}\, e^{2\pi(\omega_L+\omega_R)M}\,, \qquad (M \to \infty) \tag{5.35}$$

$$W_{\mathrm{b,f}}(\beta M) \sim \Big(\frac{\beta M}{2\pi}\Big)^{\frac{D-1}{2}} e^{-\beta M}\,, \qquad (\beta M \to \infty), \tag{5.36}$$

where ω_L and ω_R in the first line are positive numbers depending on specific models, and one can find the derivation of Eq.(5.35) in [53], while the second line is due to Eq.(A.56). This justifies the qualitative arguments made in the beginning of this section: we see the exponential growth of the degeneracies in Eq.(5.35) as well as the Boltzmann factor $e^{-M/T}$ in Eq.(5.36). For the models that we have encountered, we have [53], for heterotic string Eq.(5.20), $\big(\omega_L, \omega_R\big) = \big(\frac{\sqrt{2}}{2}, 1\big)$, and for type II string Eq.(5.22), $\big(\omega_L, \omega_R\big) = \big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\big)$. The asymptotic behaviors (5.35) and (5.36) show that the Hagedorn temperature is

$$\beta_{\mathrm{H}} = T_{\mathrm{H}}^{-1} = 2\pi\big(\omega_L + \omega_R\big) = \begin{cases} (2+\sqrt{2})\,\pi\,, & \text{heterotic}\,; \\ 2\sqrt{2}\,\pi\,, & \text{type II}\,. \end{cases} \tag{5.37}$$

The sum over mass spectrum in Eq.(5.33) diverges when $T > T_{\mathrm{H}}$.

We present briefly the situation in type I string. Obviously the Hagedorn temperature in the closed string sector is just as the type II string case. Then in the open string sector, basically the thermal one-loop amplitude also takes the form Eq.(5.33), and the counterpart of Eq.(5.35) is $\mathcal{P}(M), \mathcal{Q}(M) \sim \text{const.} \times M^{-D/2}\, e^{2\sqrt{2}\pi M}$. Thus Hagedorn singularity in the open string sector happens at the same temperature as in the closed string sector: $\beta_{\mathrm{H}} = 2\sqrt{2}\,\pi$, which turns out to be the Hagedorn temperature of type I string.

The Hagedorn divergence discussed above arises as a UV effect since the asymptotic behavior (5.36) results from the $\tau_2 \to 0$ limit of the integral in obtaining Eq.(5.33).

Tachyonic state in the spectrum and phase transition

All precedent sections describe thermal one-loop amplitude with sum of worldsheet instanton arising from $S^1(R_0)$. We now switch to the lattice sum representation, following the discussion in Sec.3.1, which offers more insight into the problem. We only investigate the maximally supersymmetric heterotic string for simplicity, while the discussion for other string models is just similar. Performing Poisson resummation to the instanton sum Eq.(5.19), we obtain

$$\sum_{\tilde{m}_0, n_0} (-1)^{a\tilde{m}_0 + bn_0 + \tilde{m}_0 n_0}\, e^{-\frac{\pi R_0^2}{\tau_2}|\tilde{m}_0 - \tau n_0|^2} = \frac{\sqrt{\tau_2}}{R_0} \sum_{m_0, n_0} (-1)^{bn_0}\, q^{\frac{1}{4}\Big[\frac{m_0 - \frac{a+n_0}{2}}{R_0} - n_0 R_0\Big]^2}\, \bar{q}^{\frac{1}{4}\Big[\frac{m_0 - \frac{a+n_0}{2}}{R_0} + n_0 R_0\Big]^2}. \tag{5.38}$$

For convenience we introduce a shorthand notation

$$\sum_{m,n} q^{\frac{1}{4}\left(\frac{m}{R}-nR\right)^2} \bar{q}^{\frac{1}{4}\left(\frac{m}{R}+nR\right)^2} := \Gamma_{m,n}(R). \tag{5.39}$$

where the subscripts on the right hand side are not to indicate a particular index, but to indicate deformations of the lattice sum. Using Eq.(5.38) to rewrite Eq.(5.20), yields an expression in $D-1$ dimensions

$$\begin{aligned} Z_{\mathrm{h}}(T) = \frac{V_{D-1}}{2(2\pi)^{D-1}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D+1}{2}}} \Big[& V_8\, \Gamma_{m_0,2n_0}(R_0) - S_8\, \Gamma_{m_0+\frac{1}{2},2n_0}(R_0) \\ & + O_8\, \Gamma_{m_0+\frac{1}{2},2n_0+1}(R_0) - C_8\, \Gamma_{m_0,2n_0+1}(R_0) \Big] \frac{\Gamma_{(d,16+d)}}{\eta^8 \bar{\eta}^{24}}. \end{aligned} \tag{5.40}$$

The sector of interest is $\eta^{-8}\bar{\eta}^{-24} O_8 \Gamma_{m_0+\frac{1}{2},2n_0+1}(R_0)$, where we observe that the two ground states, arising from string winding the Eulidean time circle, are of mass squared

$$M_O^2(R_0) = R_0^2 + \frac{1}{4R_0^2} - 3, \tag{5.41}$$

which is negative for $1 - \frac{\sqrt{2}}{2} < R_0 < 1 + \frac{\sqrt{2}}{2}$. Therefore if we gradually raise the temperature which is initially low, at a critical value corresponding to the radius $R_0 = 1 + \frac{\sqrt{2}}{2}$, the ground states become tachyonic. The contribution of one ground state to the thermal one-loop amplitude $\int_{\mathcal{F}} d^2\tau\, \tau_2^{\frac{D+1}{2}} e^{-\pi\tau_2 M_O^2}$ diverges at the integration limit $\tau_2 \to \infty$. Comparing to Eq.(5.37), this divergence arising as IR effect is just the Hagedorn singularity described before as UV effect. The same analysis can be carried out for type II strings, where we can also predict the Hagedorn temperature by the appearing of tachyonic states in the spectrum.

On general grounds, the appearing of tachyonic states is a symptom of destabilization of the vacuum, where the initial local minimum of scalar potential becomes a local maximum. Therefore the Hagedorn singularity, induced by tachyonic states in the thermal spectrum, indicates rather such an instability than a pathology of the theory. As the temperature raises as high as the Hagedorn temperature, the destabilization of thermal scalar potential should trigger a phase transition carrying the system into a new thermal vacuum [50, 54].

Interesting observation in microcanonical language regarding the heterotic Hagedorn temperature has been made in [55]. Considering the temperature β as a complex variable, the thermal one-loop amplitude (5.40) can be continued onto the whole complex plane, where Hagedorn temperature β_{H} is an isolated singularity besides many others situated in the region $\mathrm{Re}\beta < \beta_{\mathrm{H}}$. Especially, the singularity β_{H} leads to a branch cut on the complex β-plane. The discrepancy of $Z_{\mathrm{h}}(T)$ on the two sides of he branch cut tells, at high energy regime, the leading term in the density of states $\Omega(E)$ in the microcanonical ensemble description. The result shows that in case where there are noncompact directions in spacetime, the microcanonical ensemble temperature is $\beta = \frac{\partial \ln \Omega(E)}{\partial E} = \beta_{\rm H} + \mathcal{O}(E^{-1})$, which approaches $T_{\rm H}$ but can never surpass as the energy raises. Intuitively, if we try to raise temperature by injecting the more and more energy into the system, we end up with the temperature $T_{\rm H}$ but no more, because when we approach $T_{\rm H}$, more and more injected energy is converted into the internal wiggling of strings. In canonical description it is always possible to mathematically make T higher than $T_{\rm H}$, by contouring the singularity in the complex plane. However in such case the equivalence between the canonical and the microcanonical descriptions breaks down and the β on the canonical side is no longer the β on the microcanonical side. It is because with $\Omega(E) \sim e^{\beta_{\rm H} E}$, the saddle point approximation, the key step in proving the equivalence, is no longer valid when $\beta < \beta_{\rm H}$. This is the indication in microcanonical description that the system should enter into a new phase as the temperature approaches $T_{\rm H}$.

5.5 Application to ideal string gas with unbroken spacetime supersymmetry

In the following up chapters we will often consider the thermodynamics of string gas having spacetime supersymmetry. Although such cases have little phenomenological interest, our goal is however being concentrated on the non-perturbative effects and get rid of other factors that makes the problem complicated. Therefore here we set up the framework for description of such string gases, and give two examples which will be used later. We will simply use Z to denote the thermal string one-loop amplitude obtained for any model in the previous sections. Also we let $Z = Z_0 + Z_1$ with Z_0 the $T = 0$ amplitude and Z_1 the thermal part just as indicated in Eq.(5.16). Due to spacetime supersymmetry, Z_0 vanishes, while Z_1 takes the form Eq.(5.33), and the same thing is true of open strings (see the end of Sec.5.3), where now $\mathcal{P}(M) \equiv \mathcal{Q}(M) \coloneqq \mathcal{N}(M)$. Thus the one-loop amplitude becomes

$$Z = \beta^{1-D} V \sum_s \mathcal{N}(M_s) \left[W_{\rm b}(\beta M_s) + W_{\rm f}(\beta M_s) \right] = \beta^{1-D} V \sum_s \mathcal{N}(M_s)\, G(\beta M_s)\,, \tag{5.42}$$

where we refer to Eq.(5.34) for definitions of the functions $W_{\rm b,f}(\,\cdot\,)$. Here we use the index s to label the mass spectrum and each s represents a boson-fermion pair of degenerate mass. We have also defined the function

$$G(x) = W_{\rm b}(x) + W_{\rm f}(x) = 2 \sum_{\tilde{k}_0} \left(\frac{x}{2\pi |2\tilde{k}_0 + 1|} \right)^{\frac{D}{2}} K_{\frac{D}{2}}\big(x\,|2\tilde{k}_0 + 1|\big)\,. \tag{5.43}$$

Then using Eqs (A.55) and (A.56), we get the following asymptotic behaviors

$$G(x) = c_D - \frac{c_{D-2}}{4\pi}\, x^2 + \mathcal{O}(x^4) \quad (x \simeq 0)\,, \quad G(x) \sim 2 \left(\frac{x}{2\pi} \right)^{\frac{D-1}{2}} e^{-x} \quad (x \gg 1), \tag{5.44}$$

where

$$c_D = G(0) = \frac{\Gamma(\frac{D}{2})}{\pi^{\frac{D}{2}}} \sum_{\tilde{k}_0} \frac{1}{|2\tilde{k}_0 + 1|^D} . \tag{5.45}$$

The Helmholtz free energy density becomes

$$\mathcal{F} = -T^D \sum_s \mathcal{N}(M_s)\, G(\beta M_s) . \tag{5.46}$$

In cosmological application we consider low temperature regime where the temperature is much below the string scale. In such case we do not have Hagedorn instability, and the one-loop amplitude is calculable. Indeed we need not sum up all the mass spectrum in Eq.(5.42), since given the asymptotic behavior in Eq.(5.44) at $x \gg 1$, contribution from states of mass heavier than the temperature is exponentially suppressed by $e^{-M/T}$.

We also notice that due to the asymptotic behavior in Eq.(5.44) at $x \sim 0$, the function $G(x)$ reaches its maximum at $x = 0$. Therefore the free energy density Eq.(5.46) reaches its local minimum when some states become massless $M \neq 0 \to M = 0$. This can happen when there are states whose masses depend moduli fields, which can vanish at certain values of moduli. We anticipate the important fact that the local minima of $\mathcal{F}$ play crucial role in moduli stabilization since $\mathcal{F}$ plays the role of effective potential. Now we examine some specific examples.

Example I: maximally supersymmetric $SO(32)$-heterotic string gas

We consider a simple example used in [18], where we take the $SO(32)$ heterotic string compactified on a factorized torus $\prod_{\alpha=D}^{9} S^1(R_{\mathrm{h}\alpha})$ down to D-dimensional flat spacetime, here the subscript h for heterotic. The temperature is $\hat{T}_\mathrm{h} = 1/\hat{\beta}_\mathrm{h} = 1/(2\pi R_{\mathrm{h}0})$ which is much lower than the Hagedorn temperature so that the thermal one-loop amplitude is well defined. We stress once again that the hatted quantities are string frame quantities, and the internal radii, even though unhatted, are by default string frame quantities. The one-loop amplitude is computed in the appendix of [18], which is:

$$\begin{aligned} Z_\mathrm{h} = \hat{\beta}_\mathrm{h} \hat{V}_\mathrm{h} \times \hat{T}_\mathrm{h}^D \Bigg\{ & s_0 b_0\, c_D + \sum_{\alpha=D}^{9} 2 s_0 b_{-1}\, G\bigg(2\pi R_{\mathrm{h}0} \Big| \frac{1}{R_{\mathrm{h}\alpha}} - R_{\mathrm{h}\alpha} \Big| \bigg) \\ & + \sum_{\substack{A \geq 0,\, \bar{A} \geq -1,\, \vec{m},\, \vec{n} \\ A - \bar{A} = \vec{m}\cdot\vec{n} \\ (A,\vec{m},\vec{n}) \neq (0,\epsilon\vec{e}_\alpha,\epsilon\vec{e}_\alpha), \\ \forall\alpha, \forall\epsilon = -1,0,1}} s_A b_{\bar{A}}\, G\bigg(2\pi R_{\mathrm{h}0} \Big[4A + \sum_{\beta=D}^{9} \Big(\frac{m_\beta}{R_{\mathrm{h}\beta}} - n^\beta R_{\mathrm{h}\beta} \Big)^2 \Big]^{\frac{1}{2}} \bigg) \Bigg\}, \end{aligned} \tag{5.47}$$

where the function $G(\cdot)$ and c_D are as defined in Eqs (5.43) and (5.45). The coefficients $\{s_A\}$ and $\{b_{\bar{A}}\}$ arise from the expansion in powers of q and $\bar{q}$, where $A = 0, 1, 2, \ldots$ and $\bar{A} = -1, 0, 1, \ldots$.

They give the degeneracy of the corresponding oscillator level. The integers m_α and n^α label respectively the momentum and winding number along the α-th cycle of the internal torus. The first contribution in $Z_{\rm h}$ is associated to the massless states labeled by $(A,\vec{m},\vec{n}) = (0,\vec{0},\vec{0})$. They arise from the $\mathcal{N}_{10} = 1$ supergravity and $SO(32)$ super-vector multiplets in ten dimensions. The second contribution comes from modes whose masses can vanish at particular values of the internal radii. For each α, these states are labeled as $(A,\vec{m},\vec{n}) = (0,\epsilon\,\vec{e}_\alpha,\epsilon\,\vec{e}_\alpha)$, where $\epsilon = \pm1$ and $\vec{e}_\alpha$ is the unit vector in the direction α. The last line in (5.47) arises from the states which are never massless. It becomes substantial when Kaluza-Klein (winding) states become light, in the regime where some $R_{{\rm h}\alpha}$'s are large (small) compared to $2\pi R_{\rm h0}$ ($\frac{1}{2\pi R_{\rm h0}}$). All other modes, being always super heavy as compared to the temperature scale, yield exponentially suppressed contributions. The general behavior of the free energy density $\hat{\mathcal{F}}_{\rm h} := -\frac{Z_{\rm h}}{\hat\beta_{\rm h}\hat V_{\rm h}}$ is summarized as follows, and is represented in Fig.5.1:

• When all radii satisfy $|R_{{\rm h}\alpha}-1/R_{{\rm h}\alpha}| < 1/(2\pi R_{\rm h0})$, $\alpha = D,...,9$, the heterotic free energy density derived from (5.47) takes the form :

$$\hat{\mathcal{F}}_{\rm h} = -\hat{T}_{\rm h}^D\left\{s_0b_0\,c_D + \sum_{i=D}^{9} 2s_0b_{-1}\,G\left(2\pi R_{\rm h0}\left|\frac{1}{R_{{\rm h}\alpha}} - R_{{\rm h}\alpha}\right|\right) + \mathcal{O}(e^{-2\pi R_{\rm h0}})\right\}. \tag{5.48}$$

Due to the asymptotic behaviors (5.44), the states with quantum numbers $(A,\vec{m},\vec{n}) = (0,\epsilon\vec{e}_\alpha,\epsilon\vec{e}_\alpha)$ induce a local minimum of $\hat{\mathcal{F}}_{\rm h}$ at $R_{{\rm h}D} = \cdots = R_{\rm h9} = 1$. In each internal direction, there are two states of mass

$$\hat{M}_{{\rm h}\alpha} = \left|R_{{\rm h}\alpha} - R_{{\rm h}\alpha}^{-1}\right|. \tag{5.49}$$

which become massless at the self-dual point $R_{{\rm h}\alpha} = 1$. As is mentioned in Sec.3.2, they supply the non Cartan components and enhance the gauge symmetry $U(1) \to SU(2)$ in the corresponding internal direction. If we vary one of the radii $R_{{\rm h}\delta}$ while fixing all others at self-dual point $R_{{\rm h}\alpha} = 1$ for $\alpha \neq \delta$, the corresponding graph is as shown in the region I in Fig.5.1.

• In case where $R_{{\rm h}\delta} > 2\pi R_{\rm h0}$, while the rest $9-D$ remaining ones still sit at the self-dual point, the behavior of $Z_{\rm h}$ represented by the region III of the figure, the free energy density deduced from (5.47) becomes

$$\hat{\mathcal{F}}_{\rm h} = -\hat{T}_{\rm h}^D\big(s_0b_0 + (9-D)\,2s_0b_{-1}\big)\left[c_D + \sum_{m_\delta\neq0} G\Big(2\pi R_{\rm h0}\frac{|m_\delta|}{R_{{\rm h}\delta}}\Big)\right] + \mathcal{O}(e^{-2\pi R_{\rm h0}}). \tag{5.50}$$

We see that in addition to the massless supergravity and $SO(32)$ super-vector multiplets, there are also contributions coming from their Kaluza-Klein descendants, which are light since $R_{{\rm h}\delta}$ is large.

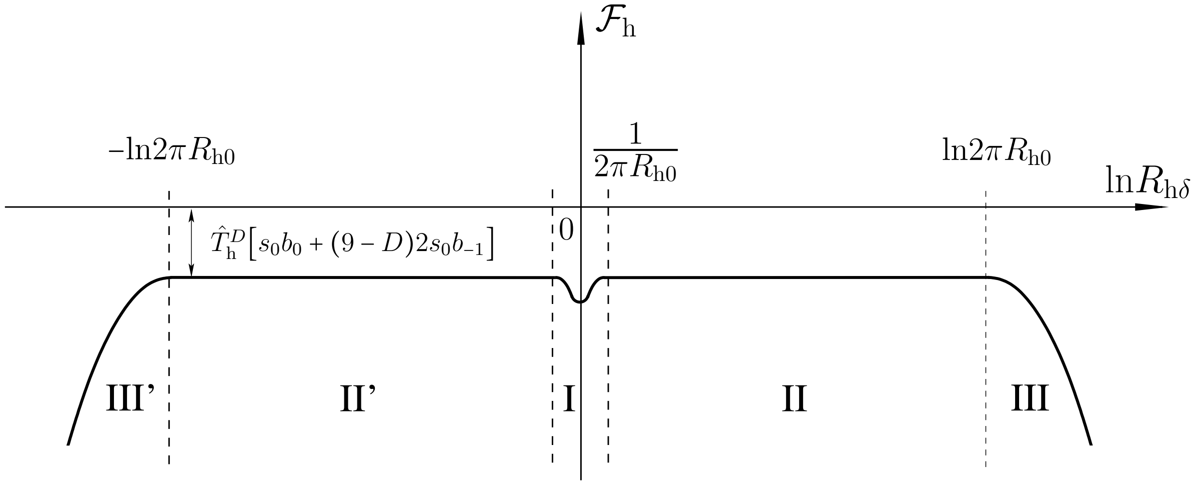


Figure 5.1: Plot of the free energy density derived from Eq.(5.47) against a given internal radius $R_{{\rm h}\delta}$, with other internal radii set to the self T-dual point 1.

• Similarly, if one radius satisfies $R_{{\rm h}\delta} < 1/(2\pi R_{\rm h0})$, while the others are set at their self-dual point, corresponding to the region III' of the figure, we have

$$\hat{\mathcal{F}}_{\rm h} = -\hat{T}_{\rm h}^D\Big(s_0 b_0 + (9-D)\,2s_0 b_{-1}\Big)\Bigg[c_D + \sum_{n_\delta\neq 0} G\Big(2\pi R_{\rm h0}|n_\delta|R_{{\rm h}\delta}\Big)\Bigg] + \mathcal{O}(e^{-2\pi R_{\rm h0}}). \tag{5.51}$$

In this case, substantial contributions arise from the winding modes along $S^1(R_{{\rm h}\delta})$, which are light since $R_{{\rm h}\delta}$ is small enough.

Example II: gas of type II string upon CY_3 compactification

The second example concerns the CY_3 compactifications of type II strings, where the CY_3 is close to some singular locus described in Sec.4.2 (conifold locus) and Sec.4.3 (non-Abelian locus). Since the models have $\mathcal{N}_4 = 2$ supersymmetry, thus in principle, Eq.(5.42) can be applied in weakly coupled regime. However in practice the exact thermal one-loop amplitude cannot be worked out since we generically do not know the whole spectrum. We are only informed of the perturbative massless states, deduced from the topology of the CY_3, as well as the light non-perturbative states from D-brane wrapping shrinking spheres. Nonetheless the knowledge of these light states suffices for us to write down the thermal one-loop amplitude at low enough temperature since contribution from heavier states are exponentially suppressed. For example in type IIA picture, we have in the perturbative spectrum, 1 gravitational multiplet, h^{11} vector multiplets and $h^{12}+1$ hypermultiplets, each multiplet containing 4 boson-fermion pairs. Furthermore in the vicinity of some singular locus, we should have additional non-perturbative light states charged under gauge groups. Inserting this mass spectrum into Eq.(5.42), we obtain

$$Z_{\mathrm{II/CY}_3} = \beta V \times T^4 \left\{ \Big[4 + 4h^{11} + 4\big(h^{12} + 1\big) \Big] G(0) + \sum_{\substack{\text{light} \\ \text{B-F pairs}}} G\big(\beta M\big) + \mathcal{O}\big(e^{-\beta M_{\min}}\big) \right\}, \tag{5.52}$$

where $M_{\min}$ is the lower mass bound of heavy states whose masses never vanish in the vicinity of the singular locus.

Chapter 6

Thermal superstring cosmology

In this chapter we formulate the thermal string cosmology scenario. The motivating idea on the technical level is to regard cosmology as solutions issue from the effective supergravity of string theories, where supersymmetry should be spontaneously broken. Whereas tree level effective supergravity accommodates only static solution with flat or AdS space background, corrections from thermal and quantum effects beyond tree level play a crucial role in generating nontrivial cosmological evolution. We consider string theories in weak coupling regime so that we compute the thermal/quantum corrections up to one-loop level. We show that the correction is implemented by adding the Colemann-Weinberg effective potential [56] to the tree level action, while the effective potential is just the free energy density of the string gas. We will also show simple applications of the formalism, and with all the results available, we will go as far as we can to illustrate out a global overview of early history of the universe. During the computation we will especially stress the aspect of moduli stabilization, which will be the theme of Chapters 7 and 8.

6.1 Setting up the scenario

The theoretical framework of thermal string cosmology is developed in a series of works [2–7,18,19] and in this section we gather the essential building blocks and give a systematic presentation. As is stated in the introduction, we consider no-scale type supergravity at tree level whose action in string frame takes the form

$$S_{\rm tree}[\hat{g},\vec{\Phi}] = \int d^Dx\sqrt{-\hat{g}}\,e^{-2\phi^{(D)}}\left[\frac{\hat{\mathcal{R}}}{2} - \frac{1}{2}\hat{\mathcal{G}}_{IJ}(\vec{\Phi})\,\partial\Phi^I\partial\Phi^J\right]. \tag{6.1}$$

We have displayed above only the bosonic action, and the hatted quantities are measured in string frame. The dilaton $\phi^{(D)}$, being one of the moduli, is included in the collective notation $\vec{\Phi}$. In cosmology, we search for homogeneous and isotropic solutions with no background scalar motion. Under such condition the equations of motion derived from the action (6.1) yield a trivial static universe, with the background space $\mathbb{R}^{1,D-1} \times M_{\text{int}}$, the product of a D-dimensional Minkowskian spacetime with some internal compact space M_{int}. It is therefore expected that nontrivial cosmological evolution can be generated when corrections induced by the thermal string gas are taken into account. Given the fact that the sting gas is nothing but the quantum fluctuations about the background, the goal is actually to consider the back reaction of quantum fluctuations on the underlying static background. We restrict ourselves to cosmological solutions, so we let the thermal/quantum induced cosmological evolution be described by the following spatially flat Robertson-Walker metric

$$d\hat{s}^2 = -\hat{N}^2(t)\, dt^2 + \hat{a}^2(t)\, d\vec{x}^2, \tag{6.2}$$

where we keep the lapse function $\hat{N}(t)$ explicit which will be useful when studying thermal background. The noncompact space is understood as a huge torus T^{D-1} of volume $\hat{V} = \hat{a}^{D-1}$, which is for the regularization of volume when doing thermodynamics. The time evolution of the metric components and all the background fields are supposed to be quasistatic, as perturbation around the static solution.

1PI effective action in thermal background

We then need to construct the quantum corrected effective action based on the tree level action (6.1), and to this end, we first refer to a field theory analogue. In field theory the back reaction of quantum fluctuations on the classical background is described by 1PI effective action. For the tree level action $S_{\text{tree}}[\phi]$ describing some field ϕ at weak coupling, the 1PI effective action $\Gamma[\phi_0]$, of some generic background configuration ϕ_0, is obtained through the path integral

$$\Gamma[\phi_0] = -\ln \int_{1\text{PI}} \mathcal{D}\eta \, \exp\Big(- S_{\text{tree}}[\phi_0 + \eta] \Big), \tag{6.3}$$

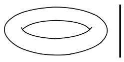
where we integrate perturbatively over the quantum fluctuations around the background ϕ_0, taking into account only the 1PI diagrams with no external legs of η. The result is an expansion in terms of η-loops:

$$\Gamma[\phi_0] = S_{\text{tree}}[\phi_0] + S_{\text{1-loop}}[\phi_0] + \dots, \tag{6.4}$$

where we omitted contribution from higher orders. The one-loop part is the sum of one-loop diagrams with all possible dispositions of external legs of ϕ_0, and the same thing should be understood to all orders in the loop expansion. Due to the ϕ_0-dependence of the η-loop diagram amplitudes, the equations of motion derived from the action (6.4) is perturbatively corrected, with respect to those derived from $S_{\text{tree}}[\phi_0]$, by effects beyond tree level. This makes the solutions of ϕ_0 that these equations yield incorporate effects from quantum effect back reaction.

In our problem of thermal string cosmology, where we search for nontrivial cosmological evolution as quantum corrections to the tree level static background, the situation is comparable. The action dictating the evolution should be a string theory analogue to the field theory 1PI effective action (6.4), which we write down diagrammatically as

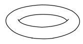
$$S_{\text{E}}[\hat{g}_{\text{E}}, \vec{\Phi}] = S_{\text{E}}^{\text{tree}}[\hat{g}_{\text{E}}, \vec{\Phi}] + S_{\text{E}}^{\text{1-loop}}[\hat{g}_{\text{E}}, \vec{\Phi}] + \dots, \tag{6.5}$$

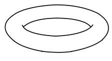
$S_{\text{E}}^{\text{1-loop}} = -$  $\Big|_{\vec{\Phi},\hat{\beta}} - \begin{pmatrix}\text{vertex operator}\\ \text{insertions}\end{pmatrix},$ (6.6)

and a similar expression including other types of worldsheet topology when open strings are considered. In obtaining Eq.(6.5), the loop amplitudes are supposed to be evaluated against a thermal background where the Euclidean time is compactified on $S^1(R_0)$ whose perimeter is the inverse temperature in string frame $\hat{\beta} = \hat{T}^{-1} = 2\pi R_0$. The subscripts "E" in Eq.(6.3) actually mean "Euclidean". The spacetime metric $\hat{g}_{\text{E}}$ appearing in Eq.(6.5) is the Wick-rotated metric of Eq.(6.2):

$$d\hat{s}_{\text{E}}^2 = \hat{\beta}^2\, dt_{\text{E}}^2 + \hat{a}^2(t_{\text{E}})\, d\vec{x}^2, \tag{6.7}$$

where the gauge choice of t_{E} is such that its metric component is $\hat{\beta}^2$.

In case where the background fields $\{\hat{a}, \vec{\Phi}\}$ are static (as the tree-level solution), the result for the one-loop part in Eq.(6.5) is merely $S_{\text{E}}^{\text{1-loop}} = -$  $\big|_{\vec{\Phi},\hat{\beta}}$, which is the thermal vacuum-to-vacuum one-loop amplitude $\ln\mathcal{Z} = Z(\hat{T})$ the central issue in Chapter 5. However here we actually consider a background $\{\hat{a}, \vec{\Phi}\}$ varying in cosmological time whose motion should be determined by the whole 1PI effective action Eq.(6.5). In such case it is necessary to consider one-loop diagrams with vertex operator insertions, which introduce kinetic corrections: $S_{\text{E}}^{\text{1-loop}} = -\ln\mathcal{Z} - \int \delta\mathcal{G}\, \partial\Phi\partial\Phi$. Here for simplicity of notation we let $\delta\mathcal{G}\partial\Phi\partial\Phi$ include also the correction to kinetic terms of the scale factor $\hat{a}$. Counting the power in string coupling $g_{\text{s}} = e^{\langle\phi^{(D)}\rangle}$, both $\ln\mathcal{Z}$ and $\delta\mathcal{G}$ are of order g_{s}^0, small compared to the tree level which is of order g_{s}^{-2}. However in our context of perturbative approach where the tree level background is static, nontrivial motions of $\hat{a}$ and $\vec{\Phi}$ are induced at least at one-loop order, making $\partial\Phi\partial\Phi$ be of order g_{s}^2. By consequence the kinetic correction $\delta\mathcal{G}\,\partial\Phi\partial\Phi$ in total is of order g_{s}^2, so that it ends up in higher order corrections than $\ln\mathcal{Z}$ and can be neglected. Therefore in the one-loop diagram in Eq.(6.5) we need only to consider vacuum-to-vacuum amplitude, so finally we are allowed to rewrite Eq.(6.5) as

$S_{\text{E}}[\hat{g}_{\text{E}}, \vec{\Phi}] = S_{\text{E}}^{\text{tree}}[\hat{g}_{\text{E}}, \vec{\Phi}] -$  $\Big|_{\vec{\Phi},\hat{\beta}} = S_{\text{E}}^{\text{tree}}[g_{\text{E}}, \vec{\Phi}] - \ln\mathcal{Z}(\hat{T}, \vec{\Phi}).$ (6.8)

Using the definition of Helmholtz free energy density $\hat{\mathcal{F}}$ (string frame) in Eq.(5.2), we have

$$\ln \mathcal{Z}(\hat{T},\vec{\Phi}) = Z(\hat{T},\vec{\Phi}) = -\hat{\beta}\hat{V}\hat{\mathcal{F}}(\hat{T},\vec{\Phi}) = -\int d^D x_{\mathrm{E}} \sqrt{\hat{g}_{\mathrm{E}}}\, \hat{\mathcal{F}}(\hat{T},\vec{\Phi}) \tag{6.9}$$

so that Eq.(6.8) becomes

$$\begin{aligned} S_{\mathrm{E}}[\hat{g}_{\mathrm{E}},\vec{\Phi}] =& S_{\mathrm{E,tree}}[\hat{g}_{\mathrm{E}},\vec{\Phi}] - Z(\hat{T},\vec{\Phi}) \\ =& \int d^D x_{\mathrm{E}} \sqrt{\hat{g}_{\mathrm{E}}} \left[e^{-2\phi^{(D)}} \left(\frac{\hat{\mathcal{R}}}{2} - \frac{1}{2}\hat{\mathcal{G}}_{IJ}\,\partial\Phi^I \partial\Phi^J \right)_{\mathrm{E}} + \hat{\mathcal{F}}(\hat{T},\vec{\Phi}) \right]. \end{aligned} \tag{6.10}$$

We stress here that $\hat{\mathcal{F}}$ is the free energy density in string frame and that the spacetime indices in the action are raised or lowered with the string frame metric $\hat{g}_{\mathrm{E}}$. This result shows that the one-loop correction induces a Coleman-Weinberg effective potential [56].

There are two remarks to make at this point. First about the reference frames, the thermal one-loop amplitude $\ln\mathcal{Z}$ being a dimensionless quantity is independent of the choice of Einstein frame or string frame. However the free energy density does, since it is of mass dimension D. More accurately let the Einstein frame free energy density be $\mathcal{F}$ and the string frame one be $\hat{\mathcal{F}}$ then $\mathcal{F} = \exp\left[\frac{2D}{D-2}\phi^{(D)}\right]\hat{\mathcal{F}}$. Second, although technically finite temperature is implemented through a Scherk-Schwarz reduction on $S^1(R_0)$, however in the tree level action no concrete Scherk-Schwarz reduction is carried out, i.e. no mass gap of order R_0^{-1} is induced. In fact the Scherk-Schwarz reduction on $S^1(R_0)$ is just a technical device which sets in only in the computation of the one-loop amplitude.

Back to Minkowskian background

To do cosmology, we rotate back to Lorentzian signature from Eq.(6.7) to Eq.(6.2), and again we mention that the background is supposed to evolve quasi-statically in the real cosmological time t. Consequently at each instant the universe is in thermal equilibrium, which allows us to quantize the underlying string theory with technique in Chapter 2 and to compute the thermal one-loop as in Chapter 5. The inverse Wick rotation yields from Eq.(6.5) the following action:

$$S[\hat{g},\vec{\Phi}] = \int d^D x \sqrt{\hat{g}} \left[e^{-2\phi^{(D)}} \left(\frac{\hat{\mathcal{R}}}{2} - \frac{1}{2}\hat{\mathcal{G}}_{IJ}\,\partial\Phi^I \partial\Phi^J \right) - \hat{\mathcal{F}}(\hat{T},\vec{\Phi}) \right]. \tag{6.11}$$

Note that the sign of $\hat{\mathcal{F}}$ should be reversed with respect to the Eulcidean action (6.10). The action in terms of Einstein frame quantities is obtained by rescaling the metric as $\hat{g} = e^{\frac{4}{D-2}\phi^{(D)}} g$.

$$S[g,\vec{\Phi}] = \int d^D x \sqrt{g} \left[\frac{\mathcal{R}}{2} - \frac{1}{2}\mathcal{G}_{IJ}\,\partial\Phi^I \partial\Phi^J - \mathcal{F}(T,\vec{\Phi}) \right], \tag{6.12}$$

where $\mathcal{F} = e^{\frac{2D}{D-2}\phi^{(D)}}\hat{\mathcal{F}}$ is the Einstein frame free energy density, and $T = 1/\beta = e^{\frac{2}{D-2}\phi^{(D)}}\hat{T}$ is the Einstein frame temperature. We observe that $\mathcal{F}$ interfere in the action as an effective potential for the moduli fields, showing that the vacuum structure is corrected at one-loop level. The Robertson-Walker metric involved in the above action is actually

$$ds^2 \equiv e^{\frac{4}{D-2}}d\hat{s}^2 := -N^2(t)\,dt^2 + a^2(t)\,d\vec{x}^2, \tag{6.13}$$

where $d\hat{s}^2$ is as in Eq.(6.2). Among all the gauge choices of the lapse function, we single out the one which makes $N = \beta$, under which the cosmological time is denoted by $\tilde{t} = \tilde{t}(t)$:

$$ds^2 = -\beta^2(\tilde{t})\,d\tilde{t}^2 + a^2(\tilde{t})\,d\vec{x}^2, \quad N(t) = \beta(\tilde{t}(t)) \times \frac{d\tilde{t}}{dt}. \tag{6.14}$$

To derive the equations of motion from the action, we refer to Appendix B, where Eqs (B.13)–(B.16) can be fit into our scenario by setting $s = 1$ and $k = 0$. Therefore we are led to the following equations of motion, where for simplicity they are written out in the gauge $N = 1$:

$$\frac{1}{2}(D-1)(D-2)H^2 - \frac{1}{2}\mathcal{G}_{MN}\dot{\Phi}^M\dot{\Phi}^N - \rho = 0, \tag{6.15}$$

$$\frac{1}{2}(D-2)\left[2\dot{H} + (D-1)H^2\right] + \frac{1}{2}\mathcal{G}_{MN}\dot{\Phi}^M\dot{\Phi}^N + P = 0, \tag{6.16}$$

$$\frac{d}{dt}\left(\mathcal{G}_{MN}\dot{\Phi}^N\right) - \frac{1}{2}\mathcal{G}_{NP,M}\dot{\Phi}^N\dot{\Phi}^P + (D-1)H\mathcal{G}_{MN}\dot{\Phi}^N + \mathcal{F}_{,M} = 0. \tag{6.17}$$

Referring to Eq.(B.12), we can read off the energy density and the pressure of the string gas

$$\rho = \left[N\frac{\partial\mathcal{F}}{\partial N} + \mathcal{F}\right]_{N=1}, \quad P = -\frac{a}{D-1}\frac{\partial\mathcal{F}}{\partial a} - \mathcal{F}\,. \tag{6.18}$$

Note that these quantities sourcing the cosmological evolution arise at one-loop level. Therefore the cosmological evolution is a pure thermal/quantum effect. We still need to verify whether the above ρ and P are the really the thermodynamic quantities . Using the second equation of Eq.(6.14), we have in the first equation $N(\partial\mathcal{F}/\partial N) = \beta(\partial\mathcal{F}/\partial\beta)$. Also provided that the $(D-1)$-dimensional space volume is $V = a^{D-1}$, we have $\partial/\partial a = (D-1)\,a^{-1}V \times \partial/\partial V$. With these observations, recalling that the free energy density is related to the partition function by Eq.(5.2), then Eq.(6.18) turn out to be

$$\rho = -\frac{\partial\ln\mathcal{Z}}{\partial\beta}, \quad P = -\frac{\partial F}{\partial V} = \frac{\partial(T\ln\mathcal{Z})}{\partial V}. \tag{6.19}$$

These are exactly the thermodynamic formulae in the canonical ensemble description Eq.(5.3). Remarkably they are obtained purely from cosmological dynamics, showing that cosmological dynamics is compatible with thermodynamics. To complete the set of equations of motion we finally have the continuity equation

$$\dot{\rho} + (D-1)H(\rho + P) = \dot{\Phi}^M \mathcal{F}_{,M}\,. \tag{6.20}$$

When $\mathcal{F}$ does not depend on a, meaning that the Helmholtz free energy is extensive, which is usually the case, we have $\mathcal{F} = -P$. Thus using $\dot{\Phi}^M P_{,M} = \dot{P} - \dot{T}\,\partial_T P$, the continuity equation becomes the conservation of entropy $S = \beta(E + PV)$:

$$0 = \frac{d}{dt}\Big[\frac{a^{D-1}(\rho + P)}{T}\Big] = \frac{d}{dt}\Big[\frac{1}{T}(E + PV)\Big] = \dot{S}\,. \tag{6.21}$$

6.2 Thermal-quantum induced cosmological evolution in supersymmetric models

We present here general features of the thermal-quantum induced cosmological evolution with a simple application. The approach to be presented is valid only in the cosmological era where the temperature T satisfies $M_{\rm s} \gg T \gg \Lambda_{\rm ew}$, which is identified as the intermediate era. The upper bound is imposed to avoid Hagedorn singularity, and the reason for lower bound will be clarified later in Sec.6.4.

Solving the equations of motion at one loop

To illustrate the cosmological evolution, we take the first example in Sec.5.5 to begin with. We study the cosmology induced by the maximally supersymmetric heterotic string gas. The low energy effective action containing the one-loop correction is

$$S_{\rm h} = \int d^D x\sqrt{-g}\left[\frac{\mathcal{R}}{2} - \frac{2}{D-2}\,\partial\phi_{\rm h}^{(D)}\partial\phi_{\rm h}^{(D)} - \frac{1}{2}\sum_{\alpha=D}^{9}\partial\left(\ln R_{{\rm h}\alpha}\right)\partial\left(\ln R_{{\rm h}\alpha}\right) - \mathcal{F}_{\rm h}(T, \phi_{\rm h}^{(D)}, R_{{\rm h}\alpha})\right] \tag{6.22}$$

where $\mathcal{F}_{\rm h}$ is the Einstein frame free energy density related to the string frame counterpart Eq.(5.47) by $\mathcal{F}_{\rm h} = e^{\frac{2D}{D-2}\phi_{\rm h}^{(D)}}\hat{\mathcal{F}}_{\rm h} = -e^{\frac{2D}{D-2}\phi_{\rm h}^{(D)}}\frac{Z_{\rm h}}{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}$. We have also the Einstein frame temperature $T = e^{\frac{2}{D-2}}\hat{T}_{\rm h}$, where there is no need to put the subscript "h". The equations of motion Eqs (6.15)–(6.20) can be applied directly, and we have

$$\frac{(D-1)(D-2)}{2}H^2 = \frac{2}{D-2}\left[\dot\phi_{\rm h}^{(D)}\right]^2 + \frac{1}{2}\sum_\alpha \dot\Phi_{{\rm h}\alpha}^2 + \rho_{\rm h}, \tag{6.23}$$

$$\ddot\phi_{\rm h}^{(D)} + (D-1)H\dot\phi_{\rm h}^{(D)} + \frac{D-2}{4}\mathcal{F}_{{\rm h},\phi} = 0, \tag{6.24}$$

$$\ddot\Phi_{{\rm h}\alpha} + (D-1)H\dot\Phi_{{\rm h}\alpha} + \mathcal{F}_{{\rm h},\alpha} = 0, \tag{6.25}$$

$$\frac{a^{D-1}}{T}(\rho_{\rm h} + P_{\rm h}) = \text{const.}, \tag{6.26}$$

where $\Phi_{{\rm h}\alpha} := \ln R_{{\rm h}\alpha}$ and the subscripts "$,\phi$" and "$,\alpha$" in the last two lines denote respectively the derivative against the dilaton and against the scalar field $\Phi_{{\rm h}\alpha}$. The equation containing second time derivative of a is omitted because it is redundant. We look for solutions at minimum of the scalar potential, which is motivated by the need of stabilizing the moduli. According to Sec.5.5, this is located at the self-dual point $R_{{\rm h}\alpha} = 1$ or $\Phi_{{\rm h}\alpha} = 0$ for all $\alpha = D, \dots, 9$. The expression of free energy density, or of the effective potential, is already give by Eq.(5.48), which in Einstein frame becomes

$$\mathcal{F}_{\rm h}(T, \phi_{\rm h}^{(D)}, R_{{\rm h},\alpha}) = -T^D\left\{s_0 b_0\, c_D + \sum_{\alpha=D}^{9} 2s_0 b_{-1}\, G\left(e^{\frac{2}{D-2}\phi_{\rm h}^{(D)}}\frac{\hat M_{{\rm h}\alpha}}{T}\right) + \mathcal{O}(e^{-2\pi R_{{\rm h}0}})\right\}, \tag{6.27}$$

where the masses $\hat M_{{\rm h}\alpha}$ are measured in string frame and are given by Eq.(5.49). At the minimum its derivatives with respect to the dilaton and to the internal radii vanish, and thus Eq.(6.24) yields $\phi_{\rm h}^{(D)}$ = const. and Eq.(6.25) is automatically satisfied. Then we use the thermodynamical relations Eq.(6.19), and find, at local minimum of $\mathcal{F}_{\rm h}$,

$$P_{\rm h} = -\mathcal{F}_{\rm h} = T^D\Big\{\big[s_0 b_0 + 2s_0 b_{-1}\,(10-D)\big]\,c_D + \mathcal{O}(e^{-2\pi R_{{\rm h}0}})\Big\}, \quad \rho_{\rm h} = (D-1)P_{\rm h}, \tag{6.28}$$

which is nothing but the Stefan-Boltzman's law of black body radiation. Plugging these into Eqs (6.23) and (6.26) we get the particular solution

$$a_*(t) \propto T_*(t)^{-1} \propto t^{2/D}, \quad H_*^2(t) \propto a_*^{-D}(t). \tag{6.29}$$

This shows that the metric evolution is of the pattern of a universe dominated by radiation. Indeed it is a radiation-dominated solution because the total energy of the universe is supplied by the massless modes in the string gas as is given in Eq.(6.28). We mention this because there is the risk that small initial background fluctuations can be amplified during cosmological evolution and end up supplying dominating contribution to the total energy. If this would happen it would upset moduli stabilization, but in the thermal string scenario this problem does not exist. We will come to this point shortly in the next subsection.

Some more comments should be added for the cases when some of the internal radii do not sit in the potential well. When some of the radii are stationary on the plateau, the region II or II' in Fig.5.1, it is obvious that the solution Eq.(6.29) is all the same, since we still have $\mathcal{F}_{\mathrm{h},\phi} = 0 = \mathcal{F}_{\mathrm{h},\alpha}$. The case where some of the radii go beyond the region II or II' and fall off the plateau requires careful computation. The result can be found in [5,6], which shows that the rolling of the internal radii down the slope accelerates the drop of the temperature, and thus accelerates the growth of the plateau to the two sides because the size of the plateau is of order $\ln(2\pi R_0)$. Finally it turns out that the plateau catches up the running away radii, which end up stationary in the region II or II', and we have once again the cosmological evolution Eq.(6.29). Therefore in any case, we have the particular solution Eq.(6.29) with each internal radius becomes stationary in the range

$$\frac{1}{R_{\mathrm{h}0}} < R_{\mathrm{h}\alpha} < R_{\mathrm{h}0}\,, \quad \text{where} \quad \alpha = D, \dots, 9. \tag{6.30}$$

The same analysis can be generalized to cases with more internal moduli switched on. We discuss this issue on a more general basis in order to apply this discussion later to other cases. Here we do not specify string model, so we will discard all subscripts "h". We let the moduli space contain the dilaton ϕ and other moduli denoted collectively by $\vec{\Phi}$. We let the model be described by Eq.(6.12), but with the dilaton factorized out, that is

$$S = \int d^D x \sqrt{-g}\left[\frac{\mathcal{R}}{2} - \frac{2}{D-2}\,\partial\phi\,\partial\phi - \frac{1}{2}\mathcal{G}_{MN}\partial\Phi^M\partial\Phi^N - \mathcal{F}\right] \tag{6.31}$$

The equations of motion are the same as Eqs (6.15)–(6.20) but with the dilaton equation factorized out, taking the form Eq.(6.24). At a certain vicinity of $\vec{\Phi} = \vec{\Phi}_*$ in the moduli space, we suppose the low energy spectrum contains n_0 massless boson-fermion pairs and n_l light boson-fermion pairs. The latter are supposed to have string frame mass $\hat{M}_s(\vec{\Phi})$ ($s = 1, \dots, n_l$) not depending on the dilaton, and vanish at $\vec{\Phi} = \vec{\Phi}_*$. The free energy density reads

$$\mathcal{F} = -T^D\left\{n_0 c_D + \sum_{s=1}^{n_l} G\left(e^{\frac{2}{D-2}\phi}\,\frac{\hat{M}_s(\vec{\Phi})}{T}\right) + \mathcal{O}\left(e^{-\frac{M_{\min}}{T}}\right)\right\}, \tag{6.32}$$

where $M_{\min}$ is the lower mass bound of the massive modes whose masses never vanish in the vicinity of $\vec{\Phi}_*$. Clearly the heterotic model Eq.(6.22) is a special case of this model. At low enough temperature, the exponentially suppressed terms can be neglected and we may derive identities for the thermal source terms at $\vec{\Phi}_*$, including the equation of state,

$$\mathcal{F}|_{\vec{\Phi}_*} = -T^D(n_0 + n_l)\,c_D\,, \quad \mathcal{F}_{,\phi}|_{\vec{\Phi}_*} = 0\,, \quad \mathcal{F}_{,M}|_{\vec{\Phi}_*} = 0\,, \quad \rho|_{\vec{\Phi}_*} = (D-1)P|_{\vec{\Phi}_*} \propto T^D. \tag{6.33}$$

Solving the equations of motion at this particular point, one obtains the particular solutions identical to those in the model (6.22) corresponding to radiation eras:

$$a_*(t) \propto \frac{1}{T_*(t)} \propto t^{2/D}\,, \qquad \phi(t) \equiv \phi_*\,, \qquad \vec{\Phi}(t) \equiv \vec{\Phi}_*\,, \tag{6.34}$$

where dilaton is a flat direction. This describes the evolution of a radiation-dominated universe. However we still need to see if this solution is stable given small initial fluctuations.

Fluctuation of the solution and moduli stabilization

We have seen that the effective potential at one loop can lift flat directions, and particular solutions where moduli fields sit in the potential well are well established. However this does not allow to conclude that the solution (6.34) is stable and that moduli are stabilized, because the expansion of the universe can amplify small initial background fluctuations and can eventually alter the behavior of the cosmological solution. Therefore we analyze the following small time-dependent deviations

$$\begin{aligned} &a(t) = a_*(t)\big(1+\epsilon_a(t)\big)\,, \quad T(t) = T_*(t)\big(1+\epsilon_T(t)\big)\,,\\ &\phi(t) = \phi_* + \epsilon_\phi(t)\,, \quad \Phi^M(t) = \Phi^M_* + \epsilon^M(t), \end{aligned} \tag{6.35}$$

which represent homogeneous and isotropic fluctuations. Denoting $H_* = \dot a_*/a_*$, the perturbation of the internal moduli equation (6.17) gives to the leading order the standard equation of scalar dynamics:

$$\ddot\epsilon^M + (D-1)H_*\,\dot\epsilon^M + \Lambda^M{}_N\,\epsilon^N = 0 \quad \text{where} \quad \Lambda^M{}_N \equiv \mathcal{G}^{ML}\big|_{\vec\Phi_*}\mathcal{F}_{,LN}\big|_{(T_*,\phi_*,\vec\Phi_*)}. \tag{6.36}$$

$\Lambda^M{}_N$ is an effective "time-dependant squared mass matrix" evaluated for the background (6.34). Since

$$\mathcal{F}_{,MN}\big|_{\vec\Phi_*} = T^{D-2}\,e^{\frac{4\phi}{D-2}}\,\frac{c_{D-2}}{4\pi}\sum_{s=1}^{n_l}\frac{\partial^2\hat M_s^2}{\partial\Phi^M\partial\Phi^N}\bigg|_{\vec\Phi_*} \tag{6.37}$$

is semi-definite positive, $\Lambda^M{}_N$ is diagonalizable with non-negative eigenvalues[1], which we denote as $\frac{4\lambda_M^2}{D^2 t^{2(D-2)/D}}$. Here we stress the peculiarity of the thermal string senario, that the scalar mass the one-loop effective potential generates is actually time dependent. Now we can solve Eq.(6.36). In the case when some λ_M's vanish, one needs to take into account quadratic terms in Eq.(6.36) (see the discussion of the dilaton equation below). In particular, this is required when moduli sit on the plateau of their thermal effective potential (see Fig.5.1). For simplicity, we analyze the most interesting case, where all internal moduli are "massive", which means all scalars oscillate in the bottom of a potential well so that $\lambda^M > 0$. Switching to a diagonal basis of perturbations $\tilde\epsilon^M$, one obtains from (6.36)

$$\tilde\epsilon^M = \frac{t^{1/D}}{\sqrt{t}}\Big[C_+^M\,J_{\frac{D-2}{4}}(\lambda_M\,t^{2/D}) + C_-^M\,J_{-\frac{D-2}{4}}(\lambda_M\,t^{2/D})\Big], \tag{6.38}$$

[1] Since the matrices $F^{-1/2}$ and $\mathcal{F}$ are (semi-)definite positive, $F^{-1/2}\mathcal{F}F^{-1/2} = F^{1/2}\Lambda F^{-1/2}$ is semi-definite positive.

where $C_\pm^M$ are integration constants and $J_\nu(\,\cdot\,)$ are Bessel functions of the first kind[2]. Using the asymptotic behavior of the Bessel function we see that these oscillations have the late time behavior

$$\tilde\epsilon^M \sim \frac{1}{\sqrt{t}} \sin\left(\lambda_M\, t^{2/D} + \text{phase}\right). \tag{6.39}$$

When t is large enough $|\tilde\epsilon^M| \ll 1$ is satisfied.

The dilaton perturbation is more involved because the dilaton potential is monotonously increasing, and becomes a flat direction only when all $\hat M_s(\Phi)$ in Eq.(6.32) vanish. We derive from Eq.(6.24) the dilaton perturbation at leading order

$$\left(a_*^{D-1}\dot\epsilon_\phi\right)^{\cdot} + a_*^{D-1}\,\frac{1}{2}\,\mathcal{F}_{,\phi MN}\big|_{(T_*,\phi_*,\vec\Phi_*)}\epsilon^M\epsilon^N = 0 \quad \text{where} \quad \mathcal{F}_{,\phi MN}\big|_{\vec\Phi_*} \equiv \frac{4}{D-2}\,\mathcal{F}_{,MN}\big|_{\vec\Phi_*}. \tag{6.40}$$

Since the constants $C_\pm^M$ are a priori of order one, we take into account the quadratic source in "massive" epsilons. Thus, $\dot\epsilon_\phi$ can be written as the sum of the general solution to its homogeneous equation, plus a particular solution to Eq. (6.40). The former is of order $1/a_*^{D-1}$ and turns out to be dominated at late times by the latter. Actually, using (6.38), the quadratic source term involves products of Bessel functions with arguments $\lambda_P\, t^{2/D}$ and $\lambda_Q\, t^{2/D}$. Integrating it once, the dominant contribution to $a_*^{D-1}\dot\epsilon_\phi$ is found to arise from "constructive interferences", i.e. quadratic terms with $\lambda_P = \lambda_Q$. This yields the asymptotic behavior,

$$\dot\epsilon_\phi \sim -\frac{C_\phi}{a_*^{D-2}} \quad\Longrightarrow\quad \epsilon_\phi \propto \frac{1}{t^{1-4/D}} \;\text{ for } D \geq 5 \quad\text{and}\quad \epsilon_\phi \propto \ln t \;\text{ for } D = 4, \tag{6.41}$$

where C_ϕ is a fully determined coefficient quadratic in $C_\pm^M$'s and positive. For $D \geq 5$, the consistency condition $|\epsilon_\phi| \ll 1$ is automatically fulfilled at late times. On the contrary, the case $D = 4$ yields formally to a logarithmically decreasing ϵ_ϕ and one may worry that the our expansions breaks down. However, the numerical solution of the full non-linear equations of motion shows that the perturbative analysis gives the correct late time behavior [18].

To analyze the evolution of the scale factor and temperature fluctuations, we expand the energy density and pressure around the background (6.34) and find from Friedmann's equation (6.15) and the continuity equation (6.21),

$$(D-1)(D-2)\,H_*\,\dot\epsilon_a = \frac{1}{2}\,\mathcal{G}_{MN}\big|_{\vec\Phi_*}\dot\epsilon^M\dot\epsilon^N + D\rho\big|_{(T_*,\phi_*,\vec\Phi_*)}\epsilon_T - \frac{D-3}{2}\,\mathcal{F}_{MN}\big|_{(T_*,\phi_*,\vec\Phi_*)}\epsilon^M\epsilon^N, \tag{6.42}$$

$$D(\epsilon_a + \epsilon_T)\,\rho\big|_{(T_*,\phi_*,\vec\Phi_*)} = \frac{D-2}{2}\,\mathcal{F}_{MN}\big|_{(T_*,\phi_*,\vec\Phi_*)}\epsilon^M\epsilon^N. \tag{6.43}$$

[2]For $D = 6$, J_{-1} should be replaced by the Bessel function of the second kind, Y_{-1}.

It is then straightforward to solve for ϵ_a, whose asymptotic behavior is again dictated by the source terms in "constructive interferences" arising from the products $\dot{\epsilon}^M\dot{\epsilon}^N$ and $\epsilon^M\epsilon^N$ in (6.42) and (6.43). The late time scaling property of ϵ_a is found to be

$$\epsilon_a \propto \frac{a_*^2}{t} \propto \frac{1}{t^{1-4/D}}, \tag{6.44}$$

which can be used in Eq. (6.43) to find

$$\epsilon_T \propto \frac{1}{t^{1-4/D}}\,(1 + \text{oscillations with constant amplitude}). \tag{6.45}$$

In (6.44) and (6.45), the coefficients of proportionality are again fully expressed in terms of the $C_\pm^M$'s. Direct numerical analysis have been carried out to the unperturbed system of differential equations in some examples [18], and the results confirm the above solutions. We then verify that the moduli are stabilized based on these solutions of fluctuations.

For $D \geq 5$ all fluctuations have damping amplitude. By consequence, the metric evolution stays the same as the particular solution (6.34), where scalar fields sitting in the potential well have damping oscillation, and the dilaton, which is a flat direction, has its fluctuation converging to zero without oscillation. These fluctuations have the energy density and the effective pressure

$$\rho_{\rm m} = \frac{1}{2}\mathcal{G}_{MN}\dot{\epsilon}^M\dot{\epsilon}^N + \frac{2}{D-2}\dot{\epsilon}_\phi^2 \sim \frac{1}{t^{3-\frac{4}{D}}} \sim a_*^{2-\frac{3}{2}D}, \quad P_{\rm m} = \rho_{\rm m}, \tag{6.46}$$

where the contribution from dilaton fluctuation scales as $t^{\frac{8}{D}-4}$ and is therefore neglected. Meanwhile the total energy density and pressure, consisting of the contribution from the string gas and the background field fluctuations, are

$$\rho_{\rm tot} = \rho_{\rm m} + \rho_{\rm rad}, \quad P_{\rm tot} = P_{\rm m} + P_{\rm rad}, \tag{6.47}$$

where $\rho_{\rm m}$ and $P_{\rm m}$ are those introduced in Eq.(6.46), while $\rho_{\rm rad}$ and $P_{\rm rad}$ are from string gas and are just ρ and P in Eq.(6.33), where we add subscript "rad" for "radiation" to emphasize that the massless modes supply substantial contribution. Therefore $\rho_{\rm rad}$ and $P_{\rm rad}$ have the behavior

$$\rho_{\rm rad} = (D-1)P_{\rm rad} \sim t^{-D/2} \sim a_*^{-D}, \tag{6.48}$$

Comparison between Eq.(6.46) and Eq.(6.48) shows that with cosmological time, the initial background fluctuations dilute. In particular $\rho_{\rm m}$ becomes negligible compared to $\rho_{\rm rad}$, $P_{\rm m}$ negligible compared to $P_{\rm rad}$, and $P_{\rm tot}/\rho_{\rm tot} \to P_{\rm rad}/\rho_{\rm rad} = D-1$ with $t \to \infty$. Therefore initial background fluctuations do not upset the particular solution Eq.(6.34). With the universe being driven to a radiation-dominated evolution, moduli $\vec{\Phi}$ are stabilized in the potential well $\vec{\Phi}_*$.

For $D = 4$, moduli $\vec{\Phi}$ still converge to $\vec{\Phi}_*$ while the dilaton value decreases logarithmically with time. However the running away behavior of the dilaton is not harmful, since it just drives the model to even weaker coupling regime. This behavior has been confirmed by numerical solution of the equations of motion. The computation of energy density stored in background fluctuations gives exactly the same result as in Eq.(6.46), but now the contribution from the dilaton fluctuation and other moduli that are of the same order. We have, for the background fluctuations and for the radiation,

$$\rho_{\rm m} = \dot{\epsilon}_\phi^2 + \frac{1}{2}\,\mathcal{G}_{MN}\dot{\epsilon}^M\dot{\epsilon}^N \sim a_*^{-4}, \quad \rho_{\rm rad} \propto a_*^{-4}. \tag{6.49}$$

Therefore with $\rho_{\rm rad} \propto \rho_{\rm m}$, background field fluctuations do not dilute with cosmological time, neither do they dominate the total energy. The universe is not radiation-dominated but the metric and the temperature still follow the pattern of a radiation-dominated evolution as the solution Eq.(6.34). It is because the fluctuations to this particular solution, Eqs (6.44) and (6.45), being of order $\mathcal{O}(1)$, results only in a constant rescaling. Therefore as the cosmological moduli problem does not arise, the background scalar fields are therefore stabilized at $\vec{\Phi}_*$. We call this situation the radiation-like cosmological evolution.

We have one more remark to make about the scalar masses. It is stated in Sec.5.5 that the effective potential develops local minima where extra massless states appear in the spectrum, but meanwhile we have shown that it is just at these points that moduli are stabilized and become massive scalars. Naively one may wonder if we have the risk of finishing up with more massless scalars when moduli are stabilized. The answer is negative. In fact the effective potential computed at one loop level cares only about the masses at tree level, and reaches local minima whenever encountering states whose tree level mass vanish. This does not prevent these states obtain a one loop level mass when the corresponding moduli are stabilized.

We would also like to mention is the moduli trapping mechanism which attracts the background scalar to specific values through dynamical process at tree level. The setup is M-theory or type II string compactified on CY_3, where with the tree level supergravity action, either one lets the background scalar take its initial value away from the local minima of the potential [57] or let it lie initially in the flat direction with a nonzero initial velocity [58]. In both cases it is shown that the expectation value of the background scalar is dynamically attracted to some locus associated to extra massless states. However this is not really address the moduli stabilization problem, since it does not really give the moduli fields mass which end up frozen in some flat direction, and moreover if the background scalar is initially stationary in the flat direction, this trapping mechanism does not function.

6.3 Non-supersymmetric vacuum

Models considered in the previous section have little phenomenological interest because of the supersymmetric vacuum at zero temperature. Although supersymmetry is spontaneously broken by finite temperature, however since the the temperature decreases in cosmological evolution, we will anyhow end up with restored supersymmetry at late time. Therefore a phenomenological viable scenario should account for the situation with broken supersymmetry at zero temperature. This problem is dealt with in the previous works [3–6] and here we just qualitatively illustrate the essential points. The models considered in those works have $\mathcal{N}_4 = 2 \longrightarrow \mathcal{N}_4 = 0$ and $\mathcal{N}_4 = 1 \longrightarrow \mathcal{N}_4 = 0$ pattern of zero temperature supersymmetry breaking. These are realized with orbifold compactification of the $E_8 \times E_8$ heterotic string, where supersymmetry is partially broken to $\mathcal{N}_4 = 2$ or $\mathcal{N}_4 = 1$ by a non-freely acting orbifold, and then the rest of the supersymmetry is spontaneously broken by a Scherk-Schwarz reduction on one or more internal circles, where the charge Q involved in the orbifold action Eq.(3.12) can contain the spacetime fermion number, the R-symmetric charge and the E_8 charge.

For simplicity and without loss of qualitative features, we first look at the cases with only one internal circle, say $S^1(R_9)$, implementing zero temperature supersymmetry breaking [4, 6], the model is characterized by the supersymmetry breaking scale $M = \frac{e^\phi}{2\pi R_9}$ and the temperature $T = \frac{e^\phi}{2\pi R_0}$ (ϕ is the dilaton in 4D), and other internal radii. Supposing M and T be much lower than the string scale while other internal radii are at about the string scale, the free energy density takes the form

$$\mathcal{F} = T^4 f\Big(\tfrac{M}{T}, \{\text{other moduli}\}\Big) = M^4 g\Big(\tfrac{M}{T}, \{\text{other moduli}\}\Big), \tag{6.50}$$

where f and g are some functions. We observe that M appears differently from other moduli which do not participate in supersymmetry breaking. This leads to the result that while moduli other than R_9 and ϕ can be attracted to particular values in the same way as in the previous section for supersymmetric models, the supersymmetry breaking scale acquires runaway behavior $M = M(t)$. More characteristics of this model can be revealed by defining the variable $z = \frac{M}{T}$. Writing down its equation of motion yields an effective potential for z. According to the choice of charge involved in the Scherk-Schwarz reduction in the internal direction R_9, the effective potential can develop a minimum at some critical value z_c, where z can be stabilized. The equations of motion yield the following particular solution

$$T(t) \propto M(t) \propto a(t)^{-1} \propto e^{3\phi(t)} \propto 1/\sqrt{t}\,, \quad (\phi \text{ is the 4D dilaton}). \tag{6.51}$$

Just as the example discussed in the last section, this is a radiation-like solution and is insensitive to the initial condition. The computation of the thermodynamical quantities of the string gas yields

$\rho_{\rm th}/P_{\rm th} \to 4$ as the universe evolves. Here the string gas does not behave like radiation, because with $R_9 \sim R_0$ the tower of KK modes along R_9 has non negligible contribution to the free energy density. For this reason we use subscript "th" standing for "thermal" instead of using "rad" as in the last section. When taking into account also the background scalar coherent motion we have the equation of state $(\rho_{\rm th}+\rho_{\rm m})/(P_{\rm th}+P_{\rm m}) \to 3$. Here $\rho_{\rm m}$ is from scalar coherent motion and $P_{\rm m}=\rho_{\rm m}$, where the leading contribution is from the motion of $M(t)$, and the contribution from other moduli not participating in the spontaneous breaking of supersymmetry is negligible. Most significantly, the result Eq.(6.51) implies the generation of the hierarchy $M \ll M_{\rm P}$, which is a crucial step leading to a realistic MSSM-type model. In case where several internal radii are involved in supersymmetry breaking, there exist solutions where all these scales drop proportionally with temperature, while the solution still represents a radiation-like universe where the state equations become

$$\frac{\rho_{\rm th}}{P_{\rm th}} \longrightarrow 3+\left(\begin{array}{c}\text{\# of internal directions}\\ \text{breaking supersymmetry}\end{array}\right), \quad \frac{\rho_{\rm th}+\rho_{\rm m}}{P_{\rm th}+P_{\rm m}} \longrightarrow 3\,, \quad \text{with} \quad t\to\infty\,. \tag{6.52}$$

Clearly the universe cannot evolve in the pattern of Eq.(6.51) forever since when M and T become very small, we will still recover supersymmetry. As is mentioned in the Introduction this evolution halts at the electroweak phase transition point where the M is expected to be stabilized at about TeV scale by infrared effects. At the stabilization of M, the moduli participating in the spontaneous supersymmetry breaking obtain mass of order $\frac{M^2}{M_{\rm P}}$ and the non participants obtain mass of order M. Meanwhile for the fermionic superpartners of these scalars the assignment of mass is reversed: the superpartner of the former acquire masses of order M and the superpartners of the latter, $\frac{M^2}{M_{\rm P}}$. At this stage, these induced masses are real constant masses, instead of the thermal masses which decrease with $M(t)$ in time. Thus the light states are expected to be the candidate of dark matter.

6.4 Panorama of thermal string cosmology

Based on the previous discussions in this chapter, the cosmological evolution in the thermal string scenario breaks into three eras. Having established the fact that the radiation-like cosmological solutions break down at early epoch where it meets with Hagedorn instability, as well as in the late epoch when temperature is at about $\Lambda_{\rm ew}$, we distinguish the Hagedorn era when $T\sim M_{\rm s}$, the intermediate era when $M_{\rm s} \gg T \gg \Lambda_{\rm ew}$ and the standard cosmology era when $T<\Lambda_{\rm ew}$. In the intermediate era, the picture of attraction to a radiation-like universe, moduli stabilization, and the runaway behavior is well established, which all the previous discussion in this chapter has

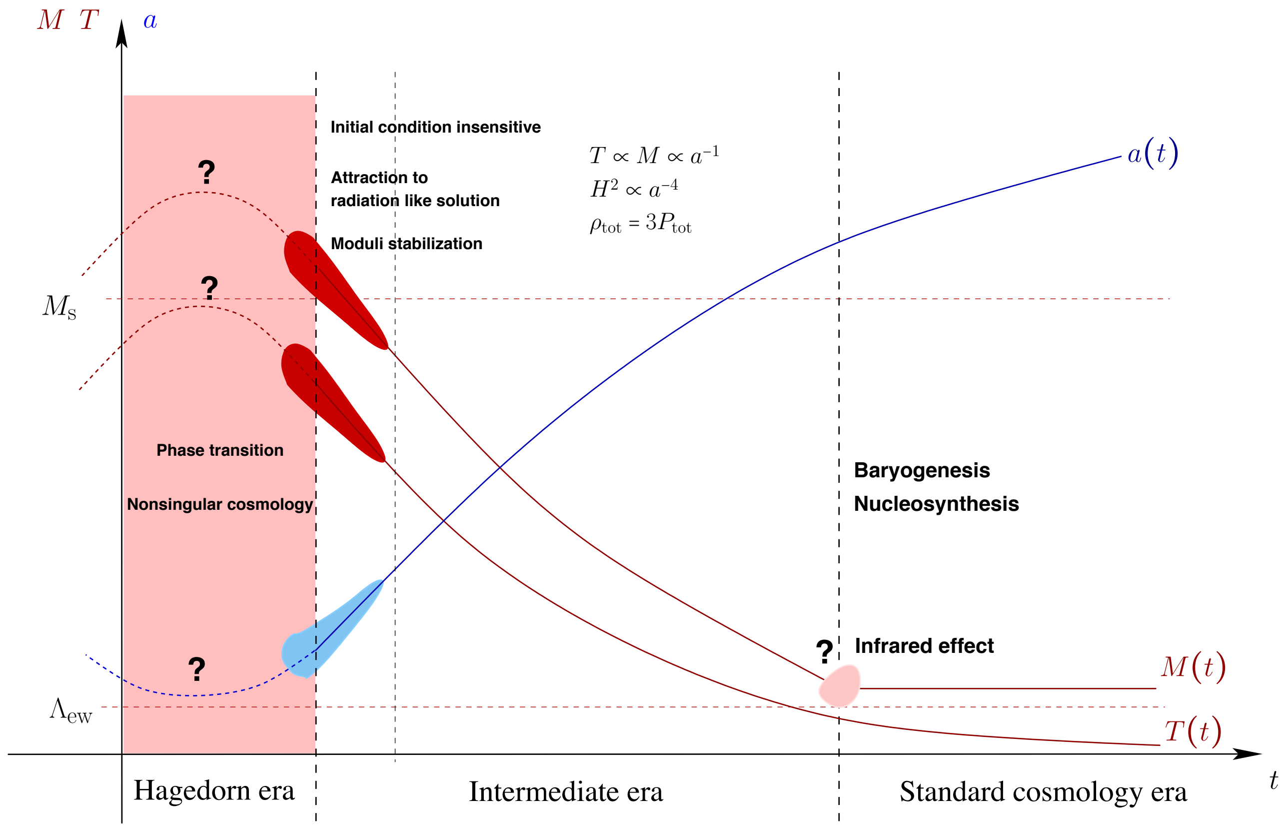


Figure 6.1: Chronology of the universe in the thermal string scenario, based on all result that we have obtained to the present day.

been devoted to. However the results in the other two eras stay more or less speculation while supporting calculation and evidence are cumulating. While some models have been suggested for resolving the Hagedorn singularity where Hagedorn phase transition can be described dynamically as the temperature goes high [11–14] leading to a nonsingular bouncing or emerging universe, the work to clarify the issue of electroweak phase transition and the stabilization of supersymmetry breaking scale is strongly expected. Assembling all results of intermediate era in Refs [2–7, 18, 19], those of Hagedorn era in Refs [11–14], as well as the speculations about the standard cosmology era based on supergravity results [9, 10], we illustrate out in Fig.6.1 as far as we can the chronology of the thermal string universe. What needs more specification is the evolution of dilaton ϕ. During the intermediate era, once zero temperature supersymmetry broken is included, $e^{3\phi}$ evolves proportionally with the temperature, as is specified in Eq.(6.51). Therefore the dilaton value can be globally small, which guarantees that the string model is always at weak coupling and the perturbative approach is always valid.

Chapter 7

Moduli stabilization in type I string by D-string states

In this chapter we explore the non-perturbative effects in the type I string which can stabilize moduli in the framework of thermal string cosmology. For simplicity, and in order to be concentrated on the non-perturbative effects, we consider maximally supersymmetric compactifications, while the results can be generalized to cases containing less supersymmetry. The type I/heterotic string-string S-duality will be used to uncover the non-perturbative effects in the type I string. The principle of this S-duality is depicted in Sec.4.1. However here it is a simple but non-trivial fact that S-duality stays valid at finite temperature. Technically, the thermal backgrounds are obtained by Scherk-Schwarz reduction with the orbifold action $(-1)^F \delta_0$, where δ_0 is an order-two shift along the Euclidean time circle and F is the fermion number. Using the "adiabatic argument" of [59], after such a free action, the two theories remain dual. Since the cosmological evolutions we study are quasi-static, it is valid to apply at each instant an S-duality transformation on the heterotic side, in order to derive non-perturbative contributions to the type I free energy and its resulting backreaction. In the following, we will first show on the heterotic side the possibility of stabilizing all moduli but the dilaton at points of gauge symmetry enhancement. The latter is due to the perturbative F-string states. Then applying S-duality maps, we show the stabilization of dual type I moduli at points where D-string states become massless and enhance gauge symmetry. Therefore cosmological evolutions induced by type I strings can lead to non-Abelian enhanced gauge symmetry by non-perturbative effects, which should be treated at equal footing as the perturbative gauge groups.

7.1 Naive perturbative type I cosmology

We have the goal of making connection with the heterotic model that we have investigated in the previous chapter. Therefore we consider the same background for the type I string with $SO(32)$ gauge group, where the internal space is the factorized torus $\prod_{i=D}^{9} S^1(R_{\mathrm{I}\alpha})$. Therefore the moduli space contains the dilaton $\phi_{\mathrm{I}}^{(D)}$ and the radii $R_{\mathrm{I}\alpha}$, and we will solve for their dynamics. We implement the temperature by compactifying the Euclidean time on a circle of perimeter $\hat{\beta}_{\mathrm{I}} = 2\pi R_{\mathrm{I}0} = 1/\hat{T}_{\mathrm{I}}$.

Working in a perturbative regime, we compute the torus, Klein-bottle, annulus and Möbius strip vacuum-to-vacuum amplitudes $\mathcal{T}$, $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$ against the thermal background. A little work following the guidelines given in Sec.5.3 yields (see appendix in [18]),

$$\mathcal{T} = \hat{\beta}_{\mathrm{I}}\hat{V}_{\mathrm{I}} \times \hat{T}_{\mathrm{I}}^{D}\left\{ s_0^2\, c_D + \sum_{\substack{A\geq 0,\, \bar{A}\geq 0,\, \vec{m},\, \vec{n} \\ A-\bar{A}=\vec{m}\cdot\vec{n} \\ (A,\vec{m},\vec{n})\neq(0,\vec{0},\vec{0})}} s_A s_{\bar{A}}\, G\!\left(2\pi R_{\mathrm{I}0}\left[4A + \sum_{\alpha=D}^{9}\left(\frac{m_\alpha}{R_{\mathrm{I}\alpha}} - n^\alpha R_{\mathrm{I}\alpha}\right)^2 \right]^{\frac{1}{2}} \right) \right\}, \tag{7.1}$$

where $\hat{V}_{\mathrm{I}}$ is the regularized volume of the $(D-1)$-dimensional space, $G(\,\cdot\,)$ and c_D are defined in Eqs (5.43) and (5.45). The integer s_A ($s_{\bar{A}}$) counts the degeneracy at oscillator level A ($\bar{A}$) on the left (right)-moving side of the worldsheet, while m_α (n^α) is the momentum (winding) number along the α-th cycle of the internal torus. The constraint $A - \bar{A} = \vec{m}\cdot\vec{n}$ is from the level matching condition. In (7.1), the first term in the braces is the contribution of the massless modes, with quantum numbers $(A,\vec{m},\vec{n}) = (0,\vec{0},\vec{0})$ and associated to the $\mathcal{N}_{10} = 1$ supergravity multiplet in ten dimensions. The Klein-bottle contribution $\mathcal{K}$ vanishes. The annulus plus Möbius amplitude takes the form

$$\mathcal{A}+\mathcal{M} = \hat{\beta}_{\mathrm{I}}\hat{V}_{\mathrm{I}} \times \hat{T}_{\mathrm{I}}^{D}\left\{ \frac{N^2-N}{2}\, s_0\, c_D + \sum_{\substack{A\geq 0,\, \vec{m} \\ (A,\vec{m})\neq(0,\vec{0})}} \frac{N^2-(-1)^A N}{2}\, s_A\, G\!\left(2\pi R_{\mathrm{I}0}\left[A + \sum_{\alpha=D}^{9}\left(\frac{m_\alpha}{R_{\mathrm{I}\alpha}}\right)^2 \right]^{\frac{1}{2}} \right) \right\}, \tag{7.2}$$

where $N = 32$ and the first term is associated to the $\mathcal{N}_{10} = 1$ $SO(32)$ super-vector multiplet in ten dimensions. The partition function is given by the sum $Z_{\mathrm{I}} = \mathcal{T} + \mathcal{K} + \mathcal{A} + \mathcal{M}$. We recall that the temperature of the system is set to be below the Hagedorn temperature. According to Sec.5.4, the Hagedorn temperature measured in string frame is $\hat{T}_{\mathrm{IH}} = (2\sqrt{2}\,\pi)^{-1}$, where the corresponding Euclidean time circle is of radius $R_{\mathrm{IH}} = \sqrt{2}$.

We can follow the lines in Sec.5.5 to figure out the behavior of the free energy density $\hat{\mathcal{F}}_{\mathrm{I}} = -Z_{\mathrm{I}}/(\hat{\beta}_{\mathrm{I}}\hat{V}_{\mathrm{I}})$. Without loss of generality, we can set free one of the internal radii $R_{\mathrm{I}\delta}$ for a given δ, while fix all the others at about the string scale. The contribution from the latter is thus exponentially suppressed, and the free energy density behaves as

$$\begin{aligned}
\hat{\mathcal{F}}_{\rm I} &= -\hat{T}_{\rm I}^{D}\left\{\left(s_0^2+\frac{N^2-N}{2}\,s_0\right)\left[c_D+\sum_{m_\delta\neq 0}G\left(2\pi R_{\rm I0}\frac{|m_\delta|}{R_{{\rm I}\delta}}\right)\right]+\mathcal{O}(e^{-2\pi R_{\rm I0}})\right\}, \quad 2\pi R_{\rm I0}<R_{{\rm I}\delta}\\
\hat{\mathcal{F}}_{\rm I} &= -\hat{T}_{\rm I}^{D}\left\{\left(s_0^2+\frac{N^2-N}{2}\,s_0\right)c_D+\mathcal{O}(e^{-2\pi R_{\rm I0}})\right\}, \quad \frac{1}{2\pi R_{\rm I0}}<R_{{\rm I}\delta}<2\pi R_{\rm I0}\\
\hat{\mathcal{F}}_{\rm I} &= -\hat{T}_{\rm I}^{D}\left\{\frac{N^2-N}{2}\,s_0\,c_D+s_0^2\left[c_D+\sum_{n_j\neq 0}G\left(2\pi R_{\rm I0}|n_\delta|R_{{\rm I}\delta}\right)\right]+\mathcal{O}(e^{-2\pi R_{\rm I0}})\right\}. \quad R_{{\rm I}\delta}<\frac{1}{2\pi R_{\rm I0}}
\end{aligned} \tag{7.3}$$

Its plot against $\ln R_{{\rm I}\delta}$ in Einstein frame is shown in Fig.7.1 by the dashed line. A difference compared to the type II and heterotic string cases, is that the open string sector is not invariant under T-duality, $R_{{\rm I}\alpha}\to 1/R_{{\rm I}\alpha}$ for any α, due to a lack of winding quantum numbers in the open sector. When $R_{{\rm I}\delta}<1$, the theory is actually better understood in the T-dual type I' picture obtained by inverting $R_{{\rm I}\delta}$.

It is straightforward to apply the analysis in Sec.6.2 to solve for the dynamics of the internal radii, from which we find again the solution Eq.(6.29) for the scale factor and the temperature in Einstein frame. The dilaton and the internal radii $R_{{\rm I}\alpha}$'s are frozen in the flat directions and the former sit in the range

$$\frac{1}{2\pi R_{\rm I0}}<R_{{\rm I}\alpha}<2\pi R_{\rm I0}\ , \quad \alpha=D,\dots,9, \tag{7.4}$$

just as in Eq.(6.30). However a very important difference is that here we have no local minimum of the free energy density where $R_{{\rm I}\delta}$, and neither is the case for the other $R_{{\rm I}\alpha}$'s, can be attracted and stabilized. However, we shall find that this type I picture obtained by purely perturbative analysis is not accurate, where we are missing important contributions from massless solitons.

7.2 D-string soliton corrected type I cosmology

Cosmology in the dual heterotic model has been investigated in Sec.6.2. Therefore we would like to find out what is missing in this naive perturbative type I cosmology from its heterotic dual.

The S-duality mapping

The type I/heterotic S-duality on general grounds are discussed in Sec.4.1. The duality maps have been given in Eqs (4.2) and (4.3), from which we derive the maps for the case being examined here with factorized internal torus:

$$\begin{aligned}
&ds^2_{\mathrm{h}(D)} = ds^2_{\mathrm{I}(D)}\\
&R_{\mathrm{h}\bar\alpha} = \frac{R_{\mathrm{I}\bar\alpha}}{\sqrt{\lambda_{\mathrm{I}}}} \equiv R_{\mathrm{I}\bar\alpha}\,\frac{e^{-\frac{1}{2}\phi_{\mathrm{I}}^{(D)}}}{\left(\prod_{\beta=D}^{9} 2\pi R_{\mathrm{I}\beta}\right)^{1/4}}\,, \quad \bar\alpha = 0 \text{ or } D,\dots,9,\\
&\phi_{\mathrm{h}}^{(D)} = -\frac{D-6}{4}\,\phi_{\mathrm{I}}^{(D)} - \frac{D-2}{8}\sum_{\alpha=D}^{9}\ln\left(2\pi R_{\mathrm{I}\alpha}\right).
\end{aligned} \tag{7.5}$$

Here the D-dimensional dilatons become $\phi_{\mathrm{h,I}}^{(D)} = \phi_{\mathrm{h,I}}^{(10)} - \frac{1}{2}\sum_{\alpha=D}^{9}\ln(2\pi R_{\mathrm{h,I}\alpha})$. The inverse maps are obtained by exchanging the subscripts $\mathrm{h} \leftrightarrow \mathrm{I}$. Note that the Euclidean radii $R_{\mathrm{I}0}$ and $R_{\mathrm{h}0}$ are included in the above relations, implying the identification of the heterotic and type I temperature in Einstein frame. Also, with the first relation we identify the the Einstein frame scale factor. Therefore we use $T = \beta^{-1}$ for the Einstein frame temperature without specifying the type of string, and the same thing for a the Einstein frame scale factor.

We let the heterotic string be at weak coupling: $e^{\phi_{\mathrm{h}}^{(D)}} \ll 1$. Therefore on the heterotic side the cosmology is determined at the one loop level, so that the result for the heterotic model in Sec.6.2 is valid. We then examine the dual type I evolution obtained by sending the heterotic result to the type I side using the maps (7.5). In other words, we forget the perturbative computation on the type I side and let its cosmology be generated by the image action of Eq.(6.22) under the S-duality maps (7.5). It is obvious that the tree level action on the two sides are identified under S-duality. However the free energy density on the heterotic side is mapped to the type I side as

$$\begin{aligned}
\mathcal{F}_{\mathrm{I}}^{\#}\big(T,\phi_{\mathrm{I}}^{(D)},R_{\mathrm{I}\alpha}\big) &= \mathcal{F}_{\mathrm{h}}\big(T,\phi_{\mathrm{h}}^{(D)},R_{\mathrm{h}\alpha}\big),\\
&= T^D\left\{s_0 b_0\, c_D + \sum_{\alpha=D}^{9} 2 s_0 b_{-1}\, G\left(2\pi R_{\mathrm{I}0}\left|\frac{1}{R_{\mathrm{I}\alpha}} - \frac{R_{\mathrm{I}\alpha}}{\lambda_{\mathrm{I}}}\right|\right)\right.\\
&\qquad\left. + \sum_{\substack{A\ge 0,\ \bar A\ge -1,\ \vec m,\ \vec n\\ A-\bar A=\vec m\cdot\vec n\\ (A,\vec m,\vec n)\neq(0,\epsilon\vec e_\alpha,\epsilon\vec e_\alpha),\\ \forall\alpha,\forall\epsilon=-1,0,1}} s_A b_{\bar A}\, G\left(2\pi R_{\mathrm{I}0}\left[\frac{4A}{\sqrt{\lambda_{\mathrm{I}}}} + \sum_{\beta=D}^{9}\left(\frac{m_\beta}{R_{\mathrm{I}\beta}} - n^\beta\frac{R_{\mathrm{I}\beta}}{\lambda_{\mathrm{I}}}\right)^2\right]^{\frac{1}{2}}\right)\right\},
\end{aligned} \tag{7.6}$$

where we put the symbol # to stress that it is different from the perturbative case result Eq.(7.3). We notice that even though the type I string can be either at strong coupling or weak coupling, the free energy density $\mathcal{F}_{\mathrm{I}}^{\#}$ always takes the form corresponding to an ideal gas. The reason is that the heterotic string is sent into type I D-string by type I/heterotic duality map, and in the current setup, the D-string is always at weak coupling even if the type I string can be strongly coupled. From type I point of view, since $\mathcal{F}_{\mathrm{I}}^{\#}$ is induced by the D-string states, thus it takes the form of the free energy density of a weakly coupled system. Especially it follows from the equality $b_0 = s_0 + (N^2 - N)/2$ (see appendix of [18]) that the massless level in $\mathcal{F}_{\mathrm{I}}^{\#}$, which is $-T^D s_0 b_0 c_D$,

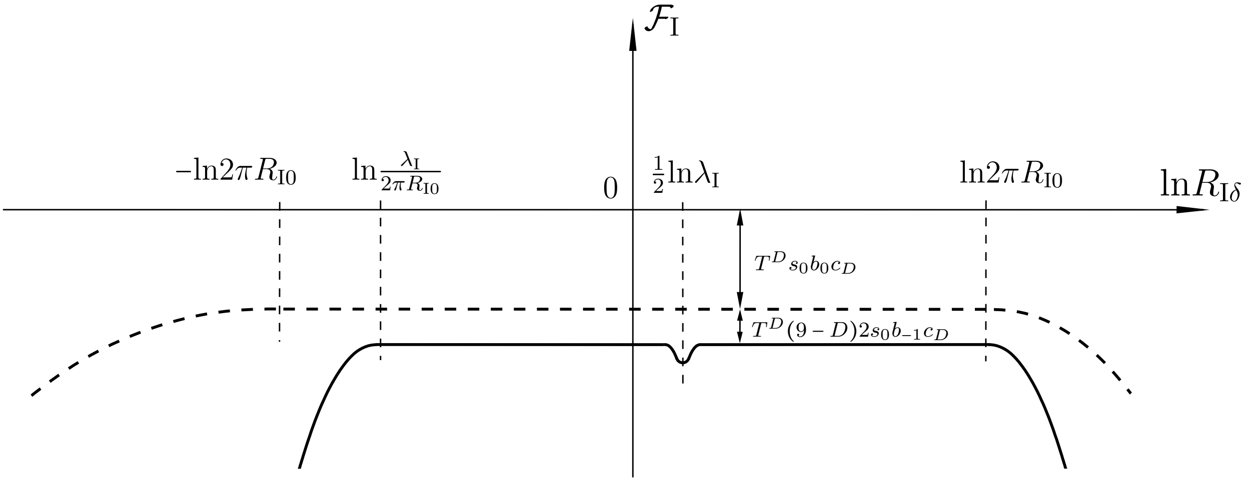


Figure 7.1: Thermal effective potential (in Einstein frame) for $R_{{\rm I}\delta}$, when all other internal radii satisfy (7.4). The dashed curve takes only into account the perturbative type I states. The solid one is obtained by heterotic/type I S-duality and receives corrections from light D-string modes.

matches the massless level in perturbative result Eqs (7.1) and (7.2) whatever the type I coupling. It is because supergravity and $SO(32)$ super-vector multiplets are short, whose mass is protected in strong coupling extrapolation.

We plot the behavior of $\mathcal{F}_{\rm I}^{\#}$ in Eq.(7.6) against one internal modulus while fixing all others at $\sqrt{\lambda_{\rm I}}$, the self T-dual point on the heterotic side, as what we did to plot Fig.5.1. The result is shown in Fig.7.1 by the solid line below the dashed line representing the result from perturbative computation. In fact it is just the image of Fig.5.1 under the maps Eq.(7.5). We will show that the discrepancy between the solid line and the dashed line is induced by the D-string state contributions.

D-string effect correction and resulting cosmology

• When all radii are around the heterotic string self T-dual point $\left|\frac{R_{{\rm I}\alpha}}{\lambda_{\rm I}}-\frac{1}{R_{{\rm I}\alpha}}\right|<\frac{1}{2\pi R_{\rm I0}}$, $\alpha=D,\dots,9$, corresponding to the region near the potential well in Fig.5.1, the type I free energy density derived from (7.6) takes the form :

$$\mathcal{F}_{\rm I}^{\#}=-T^D\left\{s_0b_0\,c_D+\sum_{\alpha=D}^{9}2s_0b_{-1}\,G\!\left(2\pi R_{\rm I0}\left|\frac{1}{R_{{\rm I}\alpha}}-\frac{R_{{\rm I}\alpha}}{\lambda_{\rm I}}\right|\right)+\mathcal{O}\big(e^{-2\pi\frac{R_{\rm I0}}{\sqrt{\lambda_{\rm I}}}}\big)\right\}. \tag{7.7}$$

The second term in the braces is of interest: it induces a local minimum at

$$R_{{\rm I}D}=\cdots=R_{{\rm I}9}=\sqrt{\lambda_{\rm I}}. \tag{7.8}$$

The states responsible for this local minimum have mass

$$\hat{M}_{\mathrm{I}\alpha} = \left| \frac{1}{R_{\mathrm{I}\alpha}} - \frac{R_{\mathrm{I}\alpha}}{\lambda_{\mathrm{I}}} \right|, \tag{7.9}$$

and these are just the S-dual states of the heterotic string states with mass given by Eq.(5.49). These states have the natural interpretation as D (or anti-D)-strings wrapped once along the circles $S^1(R_{\mathrm{I}\alpha})$, with one unit of momentum. These states lead to gauge symmetry enhancement $U(1)_\alpha \to SU(2)_\alpha$ in type I string, which is a purely non-perturbative effect. Therefore Eq.(7.7) takes the form of (massless level contribution)+(D-string soliton contribution)+(small).

Provided that we regard Eq.(7.7) as the effective scalar potential, then using the same argument as in Sec.6.2, we see that Eq.(7.8) is an attractor where these internal radii can be stabilized. Since the late time value of the string coupling in 10D satisfy $\lambda_{\mathrm{I}} = \lambda_{\mathrm{h}}^{-1} \gg 1$, the open string cosmology is well understood in type I picture rather than the T-dual type I' picture. If we denote by $\phi_{\mathrm{I}*}^{(D)}$ the asymptotic value of the type I dilaton in D dimensions , and use the inverse map of Eq.(7.5), the type I moduli converge to the following values

$$e^{\phi_{\mathrm{I}}^{(D)}(t)} \longrightarrow e^{\phi_{\mathrm{I}*}^{(D)}} \equiv \frac{e^{-\frac{D-6}{4}\phi_{\mathrm{h}*}^{(D)}}}{(2\pi)^{\frac{(10-D)(D-2)}{8}}}\,, \qquad R_{\mathrm{I}i}(t) \longrightarrow e^{\frac{2}{D-6}\phi_{\mathrm{I}*}^{(D)}}\,(2\pi)^{\frac{10-D}{D-6}} = \frac{1}{e^{\frac{1}{2}\phi_{\mathrm{h}*}^{(D)}}\,(2\pi)^{\frac{10-D}{4}}}, \tag{7.10}$$

while the temperature and scale factor asymptotic behaviors are those of a radiation dominated era, $T(t)^{-1} \sim a(t) \sim t^{2/D}$, where t is the cosmological time.

Subtleties arise when we vary the spacetime dimension. According to the S-duality map for the dilaton in Eq.(7.5), we have:

- For $D > 6$, (7.10) shows that the type I cosmology is at strong coupling, where solitons are generically light. Therefore it is natural to include the effects of states (7.9) in the low energy effective action. Moreover these D-string soliton contributions and the resulting gauge symmetry enhancement should persist at non strongly coupled regime since these states are BPS. Especially at weak coupling $\lambda_{\mathrm{I}} \ll 1$ (D-string states become heavy), we are tempted to state that the free energy density should still be given by Eq.(7.7), but with the function $G(\hat{\beta} M_\alpha)$ be replaced by some other function $\tilde{G}(\hat{\beta}, M_\alpha, \dots)$. Moreover once the soliton states become massless at the internal radius $\sqrt{\lambda_{\mathrm{I}}}$, they should give the same contribution as perturbative massless states, due to the $SU(2)$ gauge symmetry. Therefore we should have $\tilde{G}(\hat{\beta}, M_\alpha, \dots)\big|_{M_\alpha = 0} = G(0)$. In the intermediate regime $e^{\phi_{\mathrm{I}}^{(D)}} \sim 1$ we cannot figure out exactly the free energy density because Eq.(7.7) should be corrected in addition by higher string loops.

- ⋄ For $D<6$, the type I cosmological evolution is at weak coupling. However it is necessary to take into account the contributions arising from solitons which are light, when we sit in the neighborhood of the enhanced symmetry points. Additional non-perturbative states may be induced by D5-branes of the type I theory, or NS5-branes in the heterotic context, can wrap the internal manifold in analogy with the D-strings we have considered[1].

- ⋄ For $D=6$, the asymptotic values of the moduli are $e^{\phi^{(6)}_{\mathrm{I}*}}=1/(2\pi)^2$ and $R_{\mathrm{I}\alpha}(t)\to e^{-\frac{1}{2}\phi^{(6)}_{\mathrm{h}*}}/(2\pi)$. The type I picture is perturbative. Therefore for the same reason as for $D<6$ cases, the solitonic states should be counted.

In summary, for $D\neq 6$ on type I side, the internal radii are stabilized while the dilaton $\phi^{(D)}_{\mathrm{I}}$ freezes somewhere along its flat direction. On the contrary, for $D=6$, the dilaton is stabilized, all complex structures $R_{\mathrm{I}\alpha}/R_{\mathrm{I}\beta}$ are stabilized at one, while the internal space volume $\prod_{\alpha=D}^{9}(2\pi R_{\mathrm{I}\alpha})$ freezes along a flat direction. This is because in $D=6$ the heterotic/type I duality exchanges internal volumes and string couplings : $\prod_{\alpha=6}^{9}(2\pi R_{\mathrm{h,I}\alpha})\leftrightarrow 1/e^{2\phi^{(6)}_{\mathrm{I,h}}}$.

• When one of the type I internal radii satisfies $R_{\mathrm{h}\delta}>2\pi R_{\mathrm{h}0}$, while the other $9-D$ radii are stabilized, $R_{\mathrm{I}\alpha}=1$ for $\alpha\neq\delta$, the free energy density derived from (7.6) becomes

$$\mathcal{F}^{\#}_{\mathrm{I}}=-T^D\big(s_0b_0+(9-D)\,2s_0b_{-1}\big)\left[c_D+\sum_{m_\delta\neq 0}G\Big(2\pi R_{\mathrm{I}0}\frac{|m_\delta|}{R_{\mathrm{I}\delta}}\Big)\right]+\mathcal{O}(e^{-2\pi\frac{R_{\mathrm{I}0}}{\sqrt{\lambda_{\mathrm{I}}}}}).\tag{7.11}$$

This result matches that obtained from perturbative computation, the first line of (7.3), up to an additional contribution $(9-D)\,2s_0b_{-1}$ to the overall numerical coefficient. This discrepancy is given by the extra massless D (or anti-D)-strings responsible for the stabilization of the $R_{\mathrm{I}\alpha}$'s at $\sqrt{\lambda_{\mathrm{I}}}$. With respect to the pure perturbative analysis the D-string states makes plateau of the effective potential lower and the slope for $R_{\mathrm{I}\delta}>2\pi R_{\mathrm{I}0}$ steeper (see figure 7.1). The cosmological evolution is however similar to the one discussed in Sec.6.2 in the paragraph above Eq.(6.30). As their heterotic counterparts [5, 6], $R_{\mathrm{I}\delta}(t)$ and $R_{\mathrm{I}0}(t)$ evolve such that the growth of the plateau eventually catches up the internal radius rolling down the slope. After that, $R_{\mathrm{I}\delta}$ freezes along its plateau.

• When one of the dual heterotic radii satisfies $R_{\mathrm{h}\delta}<\frac{1}{2\pi R_{\mathrm{h}0}}$, which translates in type I picture into $R_{\mathrm{I}\delta}<\frac{\lambda_{\mathrm{I}}}{2\pi R_{\mathrm{I}0}}$, while the others are stabilized at their self-dual points, $R_{\mathrm{h}\alpha}=1$ for $\alpha\neq\delta$, we have

$$\mathcal{F}^{\#}_{\mathrm{I}}=-T^D\big(s_0b_0+(9-D)\,2s_0b_{-1}\big)\left[c_D+\sum_{n_\delta\neq 0}G\Big(2\pi R_{\mathrm{I}0}|n_\delta|\,\lambda_{\mathrm{I}}^{-1}R_{\mathrm{I}\delta}\Big)\right]+\mathcal{O}(e^{-2\pi\frac{R_{\mathrm{I}0}}{\sqrt{\lambda_{\mathrm{I}}}}}).\tag{7.12}$$

[1] Note that these states may contribute even for $D=5$. This is to be contrasted with 5-brane instantons at zero temperature, which require an internal space of six dimensions.

Substantial contributions arise from the modes of string frame mass

$$\hat{M}_{\mathrm{I}\delta} = |n^\delta|\frac{R_{\mathrm{I}\delta}}{\lambda_{\mathrm{I}}}, \tag{7.13}$$

which include to two sets of towers of D-string winding modes along $S^1(R_{\mathrm{I}\delta})$. The first one contains "solitonic descendants" of the perturbative massless supergravity and $SO(32)$ super-vector multiplets. Following the same logic as in the former case, we find that if at some moment $R_{\mathrm{I}\delta} < \frac{\lambda_{\mathrm{I}}}{2\pi R_{\mathrm{h}0}}$, the plateau speeds up its growth in size and prevents $R_{\mathrm{I}\delta}$ from falling down the slope. We end up in a regime where $\frac{\lambda_{\mathrm{I}}}{2\pi R_{\mathrm{h}0}} < R_{\mathrm{I}\delta}$, after which the internal modulus freezes or is stabilized at $\sqrt{\lambda_{\mathrm{I}}}$.

To finish this section, we would like to add a final remark on the Hagedorn temperature. We first observe that under the duality map (4.3), the Hagedorn radii do not match. We thus infer from the perturbative heterotic side a new value of the Hagedorn radius in type I, when λ_{I} is large:

$$R_{\mathrm{IH}} = \begin{cases} \sqrt{2} & \text{for } \lambda_{\mathrm{I}} \ll 1 \\ \sqrt{\lambda_{\mathrm{I}}}\,\frac{1+\sqrt{2}}{\sqrt{2}} & \text{for } \lambda_{\mathrm{I}} \gg 1 \end{cases}. \tag{7.14}$$

From a cosmological point of view, R_{IH} in the regime $\lambda_{\mathrm{I}}(t) \gg 1$ is thus a time-dependent scale. Note that this non-perturbative expression for R_{IH} obtained once D-strings are taken into account can be relevant even at weak coupling, $e^{\phi_{\mathrm{I}}^{(D)}} \ll 1$. This is for instance the case for $D \leq 6$, when $\sqrt{\lambda_{\mathrm{I}}}$ and the $R_{\mathrm{I}i}$'s reach the asymptotic value $\sqrt{\lambda_{\mathrm{I}0}} \gg 1$.

7.3 E1-instanton interpretation

In the previous section we have stated that D-string states can supply substantial contribution to the type I free energy density, giving rise to moduli attractors. In the Euclidean background where we compute the thermal one-loop amplitude, the D-string can be regarded as E1-brane. Especially the states responsible for moduli stabilization arise from E1-brane wrapping at once the Euclidean time circle and an internal compact direction. Therefore the non-perturbative effect revealed by type I/heterotic duality can have E1-instanton interpretation. We can clarify this issue using the technique in the work in [60] where E1-instanton contributions to holomorphic couplings are analyzed in supersymmetric cases by type I/heterotic duality. We want to infer E1 corrections in type I from dual heterotic worldsheet instantons, and for simplicity, we restrict our analysis to the case $D = 9$. This is to be contrasted with the zero temperature case where E1 corrections would only arise for $D \leq 8$.

We start on the heterotic side where the model is at finite temperature compactified on $S^1(R_9)$. We first work out the one-loop amplitude Z_{h}, expressing the lattice sum associated to the internal direction in the form of worksheet instanton just as given in the right hand side of Eq.(3.1). Thus we have in Z_h a double instanton sum, where the second one is as in Eq.(5.19) due to the finite temperature. In such case we can perform a double unfolding with respect to the internal space $S^1(R_0)\times S^1(R_9)$ [18,60], which splits the worldsheet instanton sums into the zero orbit, the degenerate orbit and the non-degenerate orbit. Accordingly, Z_h into the sum of an integration over the fundamental domaine (which vanishes due to supersymmetry), one over the strip and the third over the upper half complex plane:

$$\begin{aligned}
&Z_\mathrm{h} = Z_\mathrm{h}^\mathrm{d} + Z_\mathrm{h}^\mathrm{nd} \quad \text{with}\\
&Z_\mathrm{h}^\mathrm{d} = \frac{\hat\beta_\mathrm{h}\hat V_\mathrm{h}}{(2\pi)^9}\int_{\sqcup}\frac{d^2\tau}{2\tau_2^6}\,\frac{\Gamma_{(0,16)}}{\eta^8\bar\eta^{24}}\,R_9\sum_{\tilde m_0,\tilde m_9}{}' \, e^{-\frac{\pi R_{\mathrm{h}0}}{\tau_2}\tilde m_0^2}\, e^{-\frac{\pi R_{\mathrm{h}9}}{\tau_2}\tilde m_9^2}\Big[V_8-(-1)^{\tilde m_0}S_8\Big]\,, \qquad (7.15)\\
&Z_\mathrm{h}^\mathrm{nd} = \frac{\hat\beta_\mathrm{h}\hat V_\mathrm{h}}{(2\pi)^9}\int_{\mathbb{C}_+}\frac{d^2\tau}{2\tau_2^6}\,\frac{\Gamma_{(0,16)}}{\eta^8\bar\eta^{24}}\,R_9\,2\sum_{\substack{\tilde m_0\neq 0\\ n^9>\tilde m_9\geq 0}} e^{-\frac{\pi R_{\mathrm{h}0}}{\tau_2}\tilde m_0^2}\, e^{-\frac{\pi R_{\mathrm{h}9}}{\tau_2}|n^9\tau+\tilde m_9|^2}\Big[V_8-(-1)^{\tilde m_0}S_8\Big].
\end{aligned}$$

For the unfolding to be valid we should let $R_{\mathrm{h}9}\geq 1$, while the case $R_{\mathrm{h}9}\leq 1$ may be obtained by T-duality[2]. Performing the τ-integrations, the degenerate part Z_h^d can be brought into the form

$$Z_\mathrm{h}^\mathrm{d} = \hat\beta_\mathrm{h}\hat V_\mathrm{h}\times\hat T_\mathrm{h}^9\left\{s_0b_0\,c_9 + \sum_{A\geq 0,\,m_9}{}'\, s_Ab_A\,G\left(2\pi R_{\mathrm{h}0}\Big[4A+\Big(\frac{m_9}{R_{\mathrm{h}9}}\Big)^2\Big]^{\frac12}\right)\right\}, \qquad (7.16)$$

while the non-degenerate contribution Z_h^nd can be written as,

$$Z_\mathrm{h}^\mathrm{nd} = \hat\beta_\mathrm{h}\hat V_\mathrm{h}\times\hat T_\mathrm{h}^9\;\; 2\sum_{\substack{A\geq 0,\,\bar A\geq -1\\ n^9>\tilde m_9\geq 0}} s_Ab_{\bar A}\,\frac{e^{-2i\pi\frac{\tilde m_9}{n^9}(A-\bar A)}}{n^9}\,G\left(2\pi R_{\mathrm{h}0}\Big[4A+\Big(\frac{A-\bar A}{n^9R_{\mathrm{h}9}}-n^9R_{\mathrm{h}9}\Big)^2\Big]^{\frac12}\right). \qquad (7.17)$$

Summing over $\tilde m_9$ in (7.17) enforces the level matching condition $A-\bar A = n^9m_9$ for some integer m_9, whenever $n^9\neq 0$. Therefore $Z_\mathrm{h}^\mathrm{d}+Z_\mathrm{h}^\mathrm{nd}$ yields exactly Eq.(5.47), which can be analytically continued in the range $1\leq R_{\mathrm{h}9}\leq\sqrt{2}$. However we need to keep the sum over $\tilde m_9$ to exhibit the instantonic structure.

In Z_h^d there is only contribution from KK modes along the 0 and the 9 directions, and thus it is not informed of nontrivial instanton configuration. Therefore, it suffices to examine the non-degenerate part Z_I^nd to extract the instanton contributions. At low temperature approximation, the terms with $A\geq 1$ are at least of order $\mathcal{O}(e^{-4\pi R_{\mathrm{h}0}})$ and are exponentially suppressed compared

[2] In fact the by judging directly from the integrand one should find that the unfolding is valid for $R_{\mathrm{h}0}>\sqrt{3}$ and $R_{\mathrm{h}9}>\sqrt{2}$. The first condition is not problematic as we are focussing on the dynamics at low temperature. Actually we are interested in the stabilization of $R_{\mathrm{h}9}$ around 1, and hence the second condition could be a problem. However it can be shown that the final expression (7.17) can be analytically continued all the way to $R_{\mathrm{h}9}=1$.

to contributions from $A = 0$ level. We can define the instanton configurations by the following Kähler and complex structure moduli Υ and $\mathcal{Y}$:

$$\text{Instanton with } \; n^9 > \tilde{m}_9 \geq 0\,,\; \tilde{k}_0 \geq 0 : \begin{cases} \Upsilon = i\Upsilon_2 = i(2\tilde{k}_0+1)R_{h0}\cdot n^9 R_{h9} \\ \mathcal{Y} = \mathcal{Y}_1 + i\mathcal{Y}_2 = \dfrac{\tilde{m}_9}{n^9} + i\,\dfrac{(2\tilde{k}_0+1)R_{h0}}{n^9 R_{h9}} \end{cases}, \tag{7.18}$$

where $(2\tilde{k}_0+1)\,n^9$ is the instanton number, which counts the number of times the worldsheet wraps around the target torus. Introducing further the coefficients $\alpha_n \in \mathbb{N}$ defined in the expansion of the Bessel function in (5.43), $K_{\frac{9}{2}}(x) = \sqrt{\pi/(2x)}e^{-x}\sum_{n=0}^{4}\alpha_n/x^n$, we rewrite (7.17) as

$$Z_{\rm h}^{\rm nd} = \frac{\hat{V}_{\rm h}^{(10)}}{(2\pi)^{10}}\, 2\sum_{\rm instantons} s_0\, \frac{e^{2i\pi\Upsilon}}{\Upsilon_2\,\mathcal{Y}_2^4} \sum_{n=0}^{4}\left[\frac{\alpha_n}{(2\pi\Upsilon_2)^n}\sum_{\bar{A}\geq -1} b_{\bar{A}}\left(1+\bar{A}\,\frac{\mathcal{Y}_2}{\Upsilon_2}\right)^{4-n} e^{2i\pi\mathcal{Y}\bar{A}}\right] + c.c. + \mathcal{O}(e^{-4\pi R_{h0}}), \tag{7.19}$$

where $\hat{V}_{\rm h}^{(10)}$ is the ten-dimensional Euclidean volume. Applying the S-duality maps Eq.(4.3), the complex and Kähler structures $\mathcal{Y}$ and Υ are sent to $\mathcal{Y}_{\rm I}$ and $\Upsilon_{\rm I}/\lambda_{\rm I}$. Consequently, the exponential factor of Υ in (7.19) yields the exponential of the Nambu-Goto action for a D-string, and $Z_{\rm h}^{\rm nd}$ is mapped into a sum of E1-instantons as in [60],

$$Z_{\rm I}^{\rm E1} = \frac{\hat{V}_{\rm I}^{(10)}}{(2\pi)^{10}}\, 2\sum_{\substack{\text{E1 ins-}\\ \text{tantons}}} s_0\, \frac{e^{\frac{2i\pi}{\lambda_{\rm I}}\Upsilon_{\rm I}}}{\Upsilon_{\rm I2}\,\mathcal{Y}_{\rm I2}^4} \sum_{n=0}^{4}\left[\frac{\alpha_n}{(2\pi\Upsilon_{\rm I2})^n}\sum_{\bar{A}\geq -1} b_{\bar{A}}\left(\frac{1}{\lambda_{\rm I}}+\bar{A}\,\frac{\mathcal{Y}_{\rm I2}}{\Upsilon_{\rm I2}}\right)^{4-n} e^{2i\pi\mathcal{Y}_{\rm I}\bar{A}}\right] + c.c. + \mathcal{O}(e^{-4\pi\frac{R_{\rm I0}}{\sqrt{\lambda_{\rm I}}}}). \tag{7.20}$$

In order to reveal that the D-string contribution responsible for moduli stabilization in Eq.(7.7) issues from the above E1-instanton sum, we first notice that the dominant contribution for $A = 0$ arises when $\bar{A} = -1$ and $n^9 = 1$, while the remaining terms are exponentially suppressed. Then we recognize that the second term in Eq.(7.7) arises from the configuration with $\bar{A} = -1$, $n^9 = 1$, $\tilde{m}_9 = 0$ and $\tilde{k}_0 \geq 0$.

The result Eq.(7.20) is obtained in the strong coupling regime of the type I string $\lambda_{\rm I} \gg 1$. It will be interesting to perform a direct computation of E1-instanton correction, which is feasible at weak coupling regime $\lambda_{\rm I} \ll 1$. Recall that in the discussion made below Eq.(7.10) for $D > 6$, we speculated the weak coupling extrapolation of Eq.(7.7). We hope that the result of direct E1-instanton calculation can give further clarification to the speculation.

7.4 Type I moduli stabilization and examples

The analysis of moduli stabilization in Sec.6.2 can by all means be applied in this chapter, and in fact we can generalize to cases including all moduli arising from toroidal compactification.

On the heterotic side according to Sec.2.5, the moduli $\vec{\Phi}$ displayed in Eq.(6.31), apart from the dilaton $\phi_{\rm h}^{(D)}$, include the components of the metric $\hat{g}^{\rm h}_{\alpha\beta}$ and antisymmetric tensor $B_{\alpha\beta}$, together with the Wilson lines $Y^{\tilde{I}}_{{\rm h}\alpha}$ ($\alpha,\beta = D,\ldots,9$; $\tilde{I} = 10, 11, \ldots, 25$). Following the lines in Sec.6.2 it can be show that when the heterotic string is at weak coupling, all these heterotic moduli except the dilaton can be attracted to values of some enhanced gauge symmetry. The reason is that the perturbatively computed vacuum-to-vacuum amplitudes in heterotic theory reach local extrema at enhanced gauge symmetry points [17] whenever the enhancement leaves no $U(1)$ factor. For our case this can only induce local minima to the effective potential, giving rise to moduli attractors. This is because the extra massless states responsible for the gauge symmetry enhancement always contribute negatively to the free energy density, as is seen from Eq.(6.32). Provided that the gauge symmetry enhancement concerned here on the heterotic side is induced by F-string states, the resulting moduli stabilization is purely perturbative effect.

Moduli stabilization in the dual type I picture can be implied through the duality maps Eq.(4.3), where we need to reverse "h" and "I". The type I moduli involved include the dilaton $\phi_{\rm I}^{(D)}$, the internal metric $\hat{g}^{\rm I}_{\alpha\beta}$ the RR 2-form $C_{\alpha\beta}$, and the Wilson lines $Y^{\tilde{I}}_{{\rm I}\alpha}$. Here we make up for an subtlety that has been ignored in the previous discussions on type I theory. In Sec.7.1 we have only examined type I moduli in closed string sector and we did not find moduli attractor through perturbative calculation. However in the open string sector, perturbative states can induce moduli attractors for the Wilson line moduli. Noticing that for generic values of Wilson lines the gauge group $SO(32)$ is broken down to $U(1)^{16}$, the attractors for the Wilson lines are located at points of enhanced gauge symmetry. When the enhanced gauge group has no $U(1)$ factors, all Wilson line flat directions are lifted. Therefore in type I picture interpretation, the stabilization of all moduli is a mixture of non-perturbative D-brane effect and perturbative open string effect. The cases that needs attention is for $D = 6$ and $D = 4$. For $D = 6$ since the duality map exchanges the dilaton with the internal volume, the type I dilaton is stabilized while the internal volume is frozen at some value. For $D = 4$,the logarithmic behavior of the heterotic dilaton Eq.(6.41) is transferred to the type I dilaton.

Example : Dual heterotic/type I strings on T^2

Here we illustrate the above general analysis with examples for $D = 8$. The internal space being T^2, the moduli space of the heterotic string contains the dilaton ϕ, the Kähler modulus and the complex structure modulus of the torus $\mathcal{T} = B_{89} + i\sqrt{\hat{g}_{88}\hat{g}_{99} - \hat{g}_{89}^2}$, $\mathcal{U} = \left(\hat{g}_{89} + i\sqrt{\hat{g}_{88}\hat{g}_{99} - \hat{g}_{89}^2}\right)/\hat{g}_{88}$ and the Wilson lines $Y^{\tilde{I}}_{\alpha}$ ($\alpha = 8, 9$; $\tilde{I} = 10, 11, \ldots, 25$). Here since we work extensively on the heterotic side, and for simplicity of notation, we omit the label "h" for heterotic moduli. These translates in the type I side into the closed and open string internal moduli via the duality map $\mathcal{T} = \mathcal{T}_{\rm I}$, $\mathcal{U} = \mathcal{U}_{\rm I}$,

$Y^I_\alpha = Y^I_{\mathrm{I}\alpha}$, where

$$\mathcal{T}_{\mathrm{I}} = C_{89} + i\sqrt{\frac{\hat{g}^{\mathrm{I}}_{88}\hat{g}_{\mathrm{I}99} - \hat{g}^{\mathrm{I}\,2}_{89}}{\lambda_{\mathrm{I}}}} = C_{89} + ie^{-\phi_{\mathrm{I}}}\frac{\left(\hat{g}^{\mathrm{I}}_{88}\hat{g}^{\mathrm{I}}_{99} - \hat{g}^{\mathrm{I}\,2}_{89}\right)^{1/4}}{2\pi}\,, \quad \mathcal{U}_{\mathrm{I}} = \frac{\hat{g}^{\mathrm{I}}_{89} + i\sqrt{\hat{g}^{\mathrm{I}}_{88}\hat{g}^{\mathrm{I}}_{99} - \hat{g}^{\mathrm{I}\,2}_{89}}}{\hat{g}^{\mathrm{I}}_{88}}. \tag{7.21}$$

The only remaining flat direction of the thermal effective potential corresponds to the heterotic and type I dilatons in eight dimensions, which are related as : $\phi_{\mathrm{I}} = -\frac{1}{2}\phi - \frac{3}{4}\ln\left((2\pi)^2\sqrt{\hat{g}_{88}\hat{g}_{99} - \hat{g}^2_{89}}\right)$.

The heterotic effective action in the Einstein frame is (see for instance appendices D and E in [30])

$$S = \int d^8x\sqrt{-g}\left\{\left[\frac{\mathcal{R}}{2} - \frac{(\partial\phi)^2}{3} - \frac{1}{4}\left(\frac{|\partial\mathcal{U}|^2}{\mathcal{U}_2^2} + \frac{|\partial\mathcal{T} + Y^{\tilde{I}}_{[8}\partial Y^{\tilde{I}}_{9]}|^2}{\mathcal{T}_2^2} + \frac{|\mathcal{U}\partial Y^{\tilde{I}}_8 - \partial Y^{\tilde{I}}_9|^2}{\mathcal{T}_2\mathcal{U}_2}\right)\right] - \mathcal{F}\right\}. \tag{7.22}$$

If we arrange the thirty-four entries of the moduli vector as $\vec{\Phi} \equiv (\mathcal{T}_1, \mathcal{T}_2, \mathcal{U}_1, \mathcal{U}_2, Y^{\tilde{I}}_8, Y^{\tilde{I}'}_9)$, where indices 1 and 2 refer to real and imaginary parts, the metric components of the general expression (6.22) are

$$(F_{MN}) = \begin{pmatrix} \frac{1}{2\mathcal{T}_2^2} & 0 & 0 & 0 & -\frac{Y^{\tilde{J}}_9}{4\mathcal{T}_2^2} & \frac{Y^{\tilde{J}'}_8}{4\mathcal{T}_2^2} \\ & \frac{1}{2\mathcal{T}_2^2} & 0 & 0 & 0 & 0 \\ & & \frac{1}{2\mathcal{U}_2^2} & 0 & 0 & 0 \\ & & & \frac{1}{2\mathcal{U}_2^2} & 0 & 0 \\ & \text{sym.} & & & \frac{|\mathcal{U}|^2}{2\mathcal{T}_2\mathcal{U}_2}\delta^{\tilde{I}\tilde{J}} + \frac{Y^{\tilde{I}}_9 Y^{\tilde{J}}_9}{8\mathcal{T}_2^2} & -\frac{\mathcal{U}_1}{2\mathcal{T}_2\mathcal{U}_2}\delta^{\tilde{I}\tilde{J}'} - \frac{Y^{\tilde{I}}_9 Y^{\tilde{J}'}_8}{8\mathcal{T}_2^2} \\ & & & & & \frac{1}{2\mathcal{T}_2\mathcal{U}_2}\delta^{\tilde{I}'\tilde{J}'} + \frac{Y^{\tilde{I}'}_8 Y^{\tilde{J}'}_8}{8\mathcal{T}_2^2} \end{pmatrix}. \tag{7.23}$$

The free energy density $\mathcal{F}$ is determined by the mass spectrum as in Eq.(6.32), which is specified by the left (right)-moving oscillator level A ($\bar{A}$), the internal momenta and winding numbers m_α, n^α ($\alpha = 8, 9$), and the root vector $Q^{\tilde{I}}$ of the right-moving internal lattice $\Gamma_{\mathrm{Spin}(32)/Z_2}$. The mass formula is $\hat{M}^2_s = 2(A + \bar{A}) + \frac{1}{2}\left(p^2_L + p^2_R\right) - 2$ as in Eq.(3.6), where the internal momenta are given by Eq.(2.56). Therefore this gives the explicit mass formula

$$\hat{M}^2_{A,\vec{m},\vec{n},\vec{Q}}(\mathcal{T},\mathcal{U},Y) = \frac{1}{\mathcal{T}_2\mathcal{U}_2}\left|-m_8\mathcal{U} + m_9 + \tilde{\mathcal{T}}n^8 + \left(\tilde{\mathcal{T}}\mathcal{U} - \frac{1}{2}\mathcal{W}^{\tilde{I}}\mathcal{W}^{\tilde{I}}\right)n^9 + \mathcal{W}^{\tilde{I}}Q^{\tilde{I}}\right|^2 + 4A, \tag{7.24}$$

where we have defined

$$\mathcal{W}^{\tilde{I}} := \mathcal{U}Y^{\tilde{I}}_8 - Y^{\tilde{I}}_9\,, \qquad \tilde{\mathcal{T}} := \mathcal{T} + \frac{1}{2}Y^{\tilde{I}}_8\mathcal{W}^{\tilde{I}} \tag{7.25}$$

and used the level matching condition, $A - \bar{A} = m_\alpha n^\alpha + \frac{1}{2}Q^{\tilde{I}}Q^{\tilde{I}}$. At generic points in moduli space, the gauge group is $U(1)^2_L \times U(1)^2_R \times U(1)^{16}_R$, where $U(1)^2_L \times U(1)^2_R$ arises from T^2 compactification, and $U(1)^{16}_R$ is the Cartan subgroup of $SO(32)_R$. We now examine special points in moduli space where n_0 pairs of bosonic and fermionic superpartners generically massive are accidentally massless. Since at zero temperature the model is maximally supersymmetric, such points are associated to enhanced gauge symmetries. In fact, the additional massless modes arise at oscillator levels $A = 0$, $\bar{A} = -1$, so that n_0 is proportional to $s_0 r_{-1} = 2^3$ (see the appendix in [18]) and the enhancements of the gauge theory arise from the right-moving sector only. In the following two examples, we will simplify the notations by omitting the subscript "R" in the right-moving gauge group factors.

Local attractor 1 : $U(1)^2_L \times SU(3) \times SO(32)$

We start with the most obvious attractor where all Wilson lines vanish, $Y^I_i = 0$, leaving the $SO(32)$ group unbroken. The torus moduli take the values $\mathcal{T} = \mathcal{U} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, implying an additional $SU(3)$ gauge factor. The n_0 states responsible for the enhancement of $U(1)^2 \times U(1)^{16} \to SU(3) \times SO(32)$ are divided into two groups :

- 6×2^3 boson/fermion pairs imply $U(1)^2 \to SU(3)$. Their quantum numbers are $(\vec{m}, \vec{n}) = \pm(1,1;0,1)$, $\pm(0,1;-1,1)$ or $\pm(1,0;1,0)$, and $\vec{Q} = 0$. In this case, $p_L^{8,9} = 0 = p_R^{I\geq 10}$ and (p_R^8, p_R^9) realize the root vectors of $SU(3)_R$, which represent a hexagon.

- 480×2^3 boson/fermion pairs to recover $U(1)^{16} \to SO(32)$. They have $(\vec{m}, \vec{n}) = (\vec{0}, \vec{0})$, $\vec{Q} = \pm(1, \pm1, 0, ..., 0)$, $\pm(1, 0, \pm1, ..., 0)$ or any other permutation. In this case, $p_L^{8,9} = 0 = p_R^{8,9}$, while $(p_R^{I\geq 10})$ realize the root vectors of $SO(32)$.

To find out the scalar mass induced at one-loop level, we need to compute the squared mass matrix Λ^M_N defined in Eq. (6.36) through Eq.(6.37). The one-loop scalar masses are given by the eigenvalues of Λ^M_N.

The resulting matrix of squared masses is diagonal, with strictly positive eigenvalues [18]. Therefore, all flat directions of the internal moduli space are lifted. Reading off the eigenvalues of Λ^M_N, we find the induced one-loop mass squared are

$$M_1^2 = \frac{c_6}{4\pi}\, 2^3 \times 24\, e^{\frac{2\phi_*}{3}}\, T_*^6 \qquad \text{or} \qquad M_2^2 = \frac{c_6}{4\pi}\, 2^3 \times 240\, e^{\frac{2\phi_*}{3}}\, T_*^6\,. \tag{7.26}$$

The first one has degeneracy 4, corresponding to $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{U}_1$, $\mathcal{U}_2$, while the second is of degeneracy 32, associated to the Wilson lines $Y_8^{\tilde{I}}$ and $Y_9^{\tilde{I}}$. The additional factor of ten for the latter can be understood from the fact that they are coupled to ten times as many additional states as compared to the torus moduli.

Local attractor 2 : $U(1)^2_L \times SU(2) \times SO(34)$

The second example we consider is at the moduli values $\mathcal{T} = \mathcal{U} = i/\sqrt{2}$, $Y_8^{\tilde{I}\geq 10} = 0$ and $Y_9^{10} = -Y_9^{11} = -Y_9^{12} = \cdots = -Y_9^{25} = -1/2$. This moduli configuration is much less trivial than the previous one, since it is going to give rise to the gauge group $SU(2)_8 \times SO(34)_{9,\dots,25}$, where the subscripts denote which directions $\alpha = 8,9$ and $\tilde{I} = 10,\dots,25$ are associated with the gauge factors. There are $n_0 = 546\times 2^3$ extra massless boson/fermion pairs of states, which can be divided into 2×2^3 for the $SU(2)_8$ and 544×2^3 for the $SO(34)_{9,\dots,25}$ enhancements. Note that the $SO(34)_{9,\dots,25}$ factor arises from an enhancement of the $U(1)_9$ symmetry of the T^2 torus, with the $SO(32)$ symmetry of the internal lattice. The detailed quantum numbers of the extra states are as follows :

• 2×2^3 boson/fermion pairs give $U(1)_8 \to SU(2)_8$. They have $(\vec{m},\vec{n}) = \pm(1,0;1,0)$ and $\vec{Q} = 0$. In this case, $p_R^8 = \pm\sqrt{2}$ while other internal momenta components vanish, realizing the root vectors of $SU(2)_8$.

For $SO(34)_{9,\dots,25}$, the 544×2^3 pairs of bosons and fermions giving $U(1)^{17}_{9,\dots,25} \to SO(34)_{9,\dots,25}$ are subdivided into :

• 420×2^3 pairs transform in the adjoint representation of $SO(30)$ and are giving rise to $U(1)^{15}_{11,\dots,25} \to SO(30)_{11,\dots,25}$. 210×2^3 have $(\vec{m},\vec{n},\vec{Q}) = \pm(0,1;0,0;0,1,1,0,\dots,0)$ or any permutation of the last 15 entries. The other 210×2^3 have $(\vec{m},\vec{n},\vec{Q}) = (0,0;0,0;0,1,-1,0,\dots,0)$ or any permutation of the last 15 entries.

• 60×2^3 pairs transform as $(2,30)$ under $SO(2)_{10}\times SO(30)_{11,\dots,25}$, giving the enhanced group $SO(32)_{10,\dots,25}$. 30×2^3 of them have $(\vec{m},\vec{n},\vec{Q}) = \pm(0,1;0,0;-1,1,0,\dots,0)$ or any permutation of the last 15 entries. The other 30×2^3 have $(\vec{m},\vec{n},\vec{Q}) = \pm(0,0;0,0;1,1,0,\dots,0)$ or any permutation of the last 15 entries.

• 64×2^3 pairs transform as $(2,32)$ under $SO(2)_9\times SO(32)_{10,\dots 25}$, giving the enhanced gauge group $SO(34)_{9,\dots,25}$. 32×2^3 of them have $(\vec{m},\vec{n},\vec{Q}) = \pm(0,1;0,-1;\frac{1}{2},\dots,\frac{1}{2})$ and $\pm(0,1;0,-1;-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2})$ or any permutation of the last 15 entries. The other 32×2^3 have $(\vec{m},\vec{n},\vec{Q}) = \pm(0,2;0,-1;-\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2})$ and $\pm(0,2;0,-1;-\frac{1}{2},\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2})$ or any permutation of the last 15 entries.

Evaluation and diagonalization of the squared mass matrix in (6.36) reveals two groups of eigenvalues,

$$M_1^2 = \frac{c_6}{4\pi}\, 2^3\times 16\, e^{\frac{2\phi_*}{3}}\, T_*^6\,, \qquad\qquad M_2^2 = \frac{c_6}{4\pi}\, 2^3\times 256\, e^{\frac{2\phi_*}{3}}\, T_*^6\,. \tag{7.27}$$

The first one with degeneracy 2, is associated to $\mathcal{T}_1-\mathcal{U}_1-\frac{1}{4}(Y_8^{10}-Y_8^{11}-\dots-Y_8^{25})$ and $\mathcal{T}_2-\mathcal{U}_2$, while the second with degeneracy 34, corresponds to $\mathcal{T}_1+\mathcal{U}_1$, $\mathcal{T}_2+\mathcal{U}_2$ and all 32 Wilson lines. Thus, we find a second point in moduli space where all internal moduli are stabilized by the thermal effective potential.

Discussion

In this chapter, moduli stabilization is realized in maximally supersymmetric models. It is not hard to extend the analysis to models with supersymmetry broken at zero temperature, by taking orbifold compactifications on both sides of the type I/heterotic S-duality. We expect that the model mentioned in Sec.6.3 can be implemented in the context of this chapter. With the presence of a zero temperature supersymmetry breaking, analysis of moduli stabilization proceeds in the same way. Especially in 4D, the energy density in the background scalar oscillations becomes dominated by (instead of being proportional to) the thermal energy. However we will still have a radiation-like evolution because the motion of the supersymmetry breaking scale has non negligible contribution to the total energy density of the universe. In addition as the spacetime dimension is 4 or 5, it is possible that NS5-brane states in the heterotic string or D5-brane states in the type I string give substantial contribution, and we expect this effect can play a role in the stabilization of the dilaton. Perhaps this effect can be revealed by type II/heterotic duality in 4D.

Chapter 8

Moduli stabilization in type II Calabi-Yau compactifications

In this chapter we consider a class of models of less supersymmetry, which are type II strings compactified on Calabi-Yau three-folds. We aim to show that at finite temperature the loci corresponding to shrinking 2-spheres or 3-spheres are moduli attractors. This is due to the states from D2-brane states (in type IIA description) wrapping the shrinking spheres becoming massless at the singular loci. Since the CFT is generically unknown, we carry out the analysis based completely on the effective $\mathcal{N}_4 = 2$ supergravity action. We will show that it is possible to write down in the vicinity of such singular loci the tree level supergravity action containing all perturbative states and light soliton states. Based on the scalar potential, the tree level mass spectrum of soliton states can be obtained, where the masses turn out to be moduli dependent. In low temperature regime, this mass spectrum suffices to determine the one-loop correction, whose local minima indicate the moduli attractors. We show that this mechanism can stabilize all the Kähler moduli as well as the complex structure moduli related to the shrinking 3-cycles which can be desingularized by blowing up. However the moduli contained in the universal hypermultiplet, which are not related to CY_3 geometry, stay flat directions. An example is analyzed in the end, where the lifting of the entire Kähler moduli space is realized, with some of the complex structure moduli also stabilized. Necessary elements of Calabi-Yau compactification are reviewed in Sec.3.4, and the singular configurations of the Calabi-Yau space to be investigated are reviewed in Sec.4.2 and Sec.4.3.

8.1 Stabilization at a conifold locus

We adopt the notation in Sec.4.2. The type IIA string is compactified on a CY_3 denoted by M, of Hodge numbers h_{11} and h_{12}. Near a conifold locus, let R be respectively the number of shrinking 2-spheres and S be the number of independent homological 2-cycles that these shrinking 2-spheres represent. The conifold configuration is denoted by $\check{M}$, containing R isolated singular nodes. When $R > S$, the R nodes can be deformed into 3-spheres, desingularizing $\check{M}$ into M', where these R 3-spheres represent $R-S$ independent homological 3-cycles. The change in Hodge number is as displayed in Eq.(4.4). When $R = S$, the branch M' does not exist. However we always suppose that the branch M exist, that is, $S > 0$.

Abelian gauge theory at conifold transition locus

We sit in the branch M to specify the low energy effective supergravity action near the conifold locus. The local behavior of the action is constrained by the physics that we expect to arise at the conifold locus, as well as the $\mathcal{N}_4 = 2$ local supersymmetry.

When taking into account only perturbative field contents, the effective supergravity action should contain h_{11} massless vector multiplets and $h_{12}+1$ massless hypermultiplets in addition to the gravitational multiplet. The gauge group is $U(1)^{h_{11}+1}$, with no charged matter, so the scalar potential vanishes. Logarithmic singularity develops near the conifold locus because the initially integrated-out soliton states, from D2-brane wrapping the R vanishing 2-spheres, become massless. The singularity is repaired by integrating in these D2-brane states, represented by R hypermultiplets charged under a certain $U(1)^S$ component of the whole gauge group. We denote the space of vector multiplet scalars by $\tilde{\mathcal{M}}_{\mathrm{V}}$ and the space of hypermultiplet scalars by $\tilde{\mathcal{M}}_{\mathrm{H}}$, where we put the tildes to stress that they are different from the moduli space introduced in Sec.3.4. However they are still special Kähler manifold and quaternionic manifold respectively. The bosonic part of the repaired effective action is

$$S = \int d^4x\,\sqrt{-g}\left\{\frac{\mathcal{R}}{2} - g_{I\bar{J}}\,\partial X^I\partial\bar{X}^J - h_{\Lambda\Sigma}\nabla q^\Lambda\nabla q^\Sigma - \mathcal{V}\right\}. \tag{8.1}$$

Here $\{X^I\}$ $(I = 1,\ldots,h_{11})$ are complex coordinates of $\tilde{\mathcal{M}}_{\mathrm{V}}$ and $\{q^\Lambda\}$ $(\Lambda = 1,\ldots,4h_{12}+4R+4)$ are real coordinates of $\tilde{\mathcal{M}}_{\mathrm{H}}$. In the kinetic terms of the latter, the covariant derivative ∇ is introduced because the D2-brane induced hypermultiplets are charged under $U(1)^S$. The metrics $g_{I\bar{J}}$ and $h_{\Lambda\Sigma}$ are regular in the vicinity of the conifold locus. The gauge coupling induces a scalar potential $\mathcal{V}$.

We can choose the parameterization of $\{X^I\}$ such that X^i $(i = 1,\ldots,S)$ vanish at the conifold locus. On the type IIB side, this corresponds to the choice of setting X^i as the periods of the S

3-cycles represented by the R shrinking 3-spheres. Thus we separate these scalar fields into two groups:

$$\{X^I\} = \{X^i, X^p\}, \quad (I = 1, \ldots, h_{11};\ i = 1, \ldots, S;\ p = S+1, \ldots, h_{11}). \tag{8.2}$$

The soliton states are charged to the gauge group $U(1)^S$ associated to X^i, and X^p the rest of the scalars are spectators of conifold transition.

Also using the fact that $\tilde{\mathcal{M}}_{\mathrm{H}}$ contains a $4R$-dimensional subspace of $U(1)^S$-isometry, and the properties of quaternionic manifolds, it can be shown [19] that the real coordinates can be chosen such that

$$\{q^\Lambda\} = \{c^\alpha, q^\lambda\}, \quad (\alpha = 1, \ldots, 4R;\ \lambda = 4R+1, \ldots, 4h_{12} + 4R + 4), \tag{8.3}$$

where $\{c^\alpha\}$ parameterize the $U(1)^S$-isometric subspace, and at the conifold locus $c^\alpha = 0$. Furthermore to the lowest order expansion in c^α, the quaternionic metric has the block associated to the isometric subspace factorized out, and $\{c^\alpha\}$ can be arranged into quartets, each representing a charged hypermultiplet. More precisely, we have

$$\{c^\alpha\} = \{c^{\hat{a}u}\}, \quad (\hat{a} = 1, \ldots, R;\ u = 1, 2, 3, 4), \tag{8.4}$$

where for each quartet of a given $\hat{a}$, the four vectors $\{\partial/\partial c^{\tilde{a}1}, \partial/\partial c^{\tilde{a}2}, \partial/\partial c^{\tilde{a}3}, \partial/\partial c^{\tilde{a}4}\}$ transform among themselves under the three complex structures. The kinetic terms become

$$h_{\Lambda\Sigma} \nabla q^\Lambda \nabla q^\Sigma = \frac{1}{2} \nabla c^{\hat{a}u} \nabla c^{\hat{a}u} + h^{(0)}_{\lambda\sigma} \partial q^\lambda \partial q^\sigma + \ldots, \tag{8.5}$$

where the superscript "(0)" means the lowest order in $c^{\hat{a}u}$-expansion around 0, and the ellipses are terms of higher order in $c^{\hat{a}u}$. Let the $\hat{a}$-th hypermultiplet have charge $Q^{\hat{a}}_i$ under the i-th $U(1)$ factor of the gauge group $U(1)^S$. Thus the covariant derivative is

$$\nabla_\mu c^{\hat{a}u} = \partial_\mu c^{\hat{a}u} - Q^{\hat{a}}_i V^i_\mu c^{\hat{a}u}, \tag{8.6}$$

where V^i_μ is the vector field associated to X^i in the same vector multiplet.

With all these approximations we can write down the bosonic effective action more explicitly near the conifold locus using the $\mathcal{N}_4 = 2$ supergravity formalism (c.f. for example [38]). The resulting action contains a scalar potential due to the gauging:

$$\begin{aligned} S = \int d^4x \sqrt{-g} \Big\{ & \frac{\mathcal{R}}{2} - g^{(0)}_{I\bar{J}} \partial_\mu X^I \partial^\mu \bar{X}^J - \frac{1}{2} \nabla c^{\mathcal{A}u} \nabla^\mu c^{\mathcal{A}u} - h^{(0)}_{\lambda\sigma} \partial q^\lambda \partial q^\sigma \\ & - e^{\mathcal{K}^{(0)}_{\mathrm{V}}} \Big(2\, Q^{\hat{a}}_i Q^{\hat{a}}_j \bar{X}^i X^j \mathscr{C}^{\hat{a}\dagger} \mathscr{C}^{\hat{a}} + \frac{1}{4} g^{(0)i\bar{j}} \vec{D}_i \cdot \vec{D}_j \Big) + \ldots \Big\}. \end{aligned} \tag{8.7}$$

The superscript "(0)" indicates that the quantity is obtained by setting $X^i = 0$ and $c^{\hat{a}u} = 0$. The second line is the scalar potential, where $\mathcal{K}_{\rm V}$ is the Kähler potential of $\tilde{\mathcal{M}}_{\rm V}$, which can depend on the spectator scalars $\{X^p\}$. For simplicity of notation we have introduced the D-term and the $SU(2)_{\mathcal{R}}$ doublets, which are

$$\vec{D}_i = \sum_{\hat{a}} Q_i^{\hat{a}} \mathscr{C}^{\hat{a}\dagger} \vec{\sigma} \mathscr{C}^{\hat{a}} \,; \quad \mathscr{C}^{\hat{a}} = \begin{pmatrix} i\big(c^{\hat{a}1} + i\, c^{\hat{a}2}\big) \\ \big(c^{\hat{a}3} + i\, c^{\hat{a}4}\big)^* \end{pmatrix}. \tag{8.8}$$

It is interesting to note that the above action is that of the rigid $\mathcal{N} = 2$ supersymmetric Abelian gauge theory with charged hypermultiplets and formally coupled to gravity.

Lifting the flat directions at one-loop

The scalar potential in the second line of Eq.(8.7), valid around the conifold locus, admits flat directions which are parameterized by

$$X^i Q_i^{\hat{a}} \mathscr{C}^{\hat{a}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ (no sum over } \hat{a}) \text{ and } \vec{D}_i = 0, \quad (\hat{a} = 1, \dots, R;\ i = 1, \dots, S). \tag{8.9}$$

The conifold locus lies on these flat directions, defined by $(X^i = 0, c^{\hat{a}u} = 0; X^p, q^\alpha$ arbitrary). We can let the scalars X^i and $c^{\hat{a}u}$ move away from the locus in different ways in the flat directions given by (8.9), carrying the effective action into the Coulomb branch or the Higgs branch, and consequently the internal manifold is desingularized into M or M'. Here we recall that the Higgs branch, or the desingularization into M' exists only for $R > S$.

• Coulomb branch

The Coulomb branch of vacua corresponds to arbitrary values for the gauged vector multiplet scalars and vanishing VEV's for those in the charged hypermultiplets:

$$\text{Coulomb branch :} \quad \Big\{\big(X^i \text{ arbitrary}, c^{\hat{a}u} = 0\big)\Big\} \times \Big\{\big(X^p, q^\alpha\big) \text{ arbitrary}\Big\}. \tag{8.10}$$

Following this assignment of VEV's we have, besides the spectator massless fields, S massless vector fields and R massive hypermultiplets. Table 8.1 summarizes the superfield content and associated scalar VEV's in the Coulomb branch[1]. The latter are just the D2-brane states. We

[1]See [61] for a general discussion about field contents following a Higgs mechanism in $\mathcal{N} = 2$ supersymmetric gauge theories.

evaluate the Colemann-Weinberg effective action, based on the tree-level action (8.7) against this background, and obtain

$$S_{\text{1-loop}} = \int d^4x\,\sqrt{-g}\left\{\frac{\mathcal{R}}{2} - g_{I\bar{J}}\,\partial_\mu X^I\partial^\mu\bar{X}^J - h_{\alpha\beta}\,\partial_\mu q^\alpha\partial^\mu q^\beta - \mathcal{F} + \cdots\right\}. \tag{8.11}$$

The method of calculating the one-loop effective potential is given in Sec.5.5 in the second example. Referring to Eq.(5.52) the result is

$$\mathcal{F} = -T^4\left\{\Big[4 + 4h_{11} + 4(h_{12}+1)\Big]G(0) + 4\sum_{\hat{a}} G\Big(\frac{M_{\hat{a}}}{T}\Big) + \mathcal{O}\big(e^{-\frac{M_{\min}}{T}}\big)\right\}, \tag{8.12}$$

We stress again that the computation is valid because the gauge theory is weakly coupled near the conifold locus. The masses $M_{\hat{a}}$ in Eq.(8.12) of the R light hypermultiplets are given by

$$M_{\hat{a}}^2 = 4\,e^{\mathcal{K}_{\text{V}}^{(0)}}|Q_i^{\hat{a}}X^i|^2 + \cdots, \tag{8.13}$$

which is obtained by expanding the scalar potential about the vacuum (8.10). The dots stand for higher order terms in vanishing scalar fields. Near the conifold locus all other masses are bounded from below and heavier than the charged black holes and we denote the lower bound $M_{\min}$. All contributions $G(M_s/T)$ with $M_s \geq M_{\min}$ are exponentially suppressed, provided the temperature is low enough, $T < M_{\min}$, as indicated in Eq.(5.52). Eq.(8.13) is consistent with the standard mass formula of BPS black holes [20,62]. In particular, it acquires a dilaton dressing $e^{-\phi}$ once measured in string frame, as expected for D-brane masses.

The behavior of the G-function (5.44) shows that $\mathcal{F}$ reaches local minimum when all masses $M_{\hat{a}}$ vanish. According to Eq.(8.13), and due to the fact that the matrix $Q_i^{\hat{a}}$ is of rank S,[2], this happens only at the conifold locus when $X^i = 0$ for all $i = 1,\dots,S$. Thus all classically flat directions X^i of the Coulomb branch are lifted, while the spectator scalars X^p and q^α parameterizing the conifold locus remain moduli. Then we use Eqs (6.36) and (6.37) to compute the the masses $M_i^{(\text{1-loop})}$ that the fields X^i obtain at the potential minimum. Therefore we consider the squared mass matrix

$$\Lambda^{\bar{I}}{}_{\bar{J}} = g^{(0)\bar{I}K}\,\frac{\partial^2\mathcal{F}}{\partial X^K\partial\bar{X}^J}\bigg|_{X^i=0} = \frac{T^2}{16}\,g^{(0)\bar{I}K}\,4\sum_{\hat{a}}\frac{\partial^2 M_{\hat{a}}^2}{\partial X^K\partial\bar{X}^J}\bigg|_{X^i=0}, \tag{8.14}$$

which satisfies

$$\Lambda^{\bar{I}}{}_{\bar{j}} = T^2 e^{\mathcal{K}^{(0)}} g^{(0)\bar{I}k}\,Q_k^{\hat{a}}Q_j^{\hat{a}}\,, \qquad \Lambda^{\bar{I}}{}_{\bar{m}} = 0. \tag{8.15}$$

[2]Otherwise, some of the R vanishing 2-cycles would be linear combinations of the others and would not give independent degrees of freedom once wrapped with D2-branes.

Following from the fact that $\{Q_i^{\hat{a}}\}$ is a matrix of rank S, one can show that Λ is diagonalizable with S strictly positive eigenvalues $\big(M_i^{(1\text{-loop})}\big)^2$ and $h_{11}-S$ vanishing ones. The trace of Λ leads to

$$\sum_i \big(M_i^{(1\text{-loop})}\big)^2 = T^2 e^{\mathcal{K}_{\rm V}^{(0)}} g^{(0)\bar{\jmath}k}\, Q_k^{\hat{a}} Q_j^{\hat{a}}. \tag{8.16}$$

Thus, the scalars of the $U(1)^S$ vector multiplets acquire one-loop masses of order the temperature scale, while the gauge bosons remain massless and the full Abelian gauge theory $U(1)^{h_{11}+1}$ is unbroken.

- **Higgs branch**

Now we suppose $R > S$ and move to the Higgs branch of the $U(1)^S$ gauge theory, and we will show that the flat directions are lifted by the one-loop thermal effective potential. In this phase, the doublets $\mathscr{C}^{\hat{a}}$ are such that the D-terms in Eq.(8.9) vanish, while the $U(1)^S$ vector multiplet scalars have trivial VEV's,

$$\text{Higgs branch :} \quad \Big\{\big(X^i = 0, \mathscr{C}^{\hat{a}} \text{ such that } \vec{D}_i = 0\big)\Big\} \times \Big\{\big(X^p, q^\alpha\big) \text{ arbitrary}\Big\}. \tag{8.17}$$

The $3S$ D-term constraints leave $4R-3S$ flat directions among the charged scalars $c^{\hat{a}u}$'s, along which the $U(1)^S$ gauge group is Higgsed. S of the $4R-3S$ directions are $U(1)^S$-gauge orbits corresponding to physically equivalent vacua. Introducing S gauge fixing conditions, we are thus left with $4(R-S)$ flat directions, which can be arranged in $R-S$ massless neutral hypermultiplets. This shows that $R > S$ must be required for the Higgs branch to exist. Moreover, the S Higgsed vector multiplets become massive and long by combining with the remaining S hypermultiplets. The superfield content and VEV's in the Higgs branch can be found in Table 8.1. Thus, besides the supergravity multiplet, the massless spectrum includes $h_{11}-S$ vector multiplets and $h_{12}+R-S+1$ neutral hypermultiplets, corresponding to the type IIA compactification on the smooth CY_3 M', with Hodge numbers h'_{11} and h'_{12} given in Eq.(4.4).

To describe the one-loop effective action in the Higgs branch, it is convenient to parameterize the D-term flat directions with some coordinates ξ^m $(m = 1,\dots,4(R-S))$ satisfying $Q_i^{\hat{a}}\, \mathscr{C}^{\hat{a}\dagger}(\xi)\,\vec{\sigma}\,\mathscr{C}^{\hat{a}}(\xi) = 0$ and such that the Jacobian matrix $\big(\frac{\partial c^{\hat{a}u}}{\partial \xi^m}\big)$ is of rank $4(R-S)$. We denote by ξ_0^m the origin of the Higgs branch, i.e. the conifold locus. In a neighborhood of the conifold locus, the one-loop effective action valid in the Higgs branch takes the form,

$$S_{1\text{-loop}} = \int d^4x\,\sqrt{-g}\left\{\frac{\mathcal{R}}{2} - g^{(0)}_{p\bar{q}}\,\partial X^p \partial \bar{X}^q - h^{(0)}_{mn}\,\partial\xi^m\partial\xi^n - h^{(0)}_{\alpha\beta}\,\partial q^\alpha \partial q^\beta - \mathcal{F}\right\}, \tag{8.18}$$

where we have defined

$$h^{(0)}_{mn} = \frac{1}{2}\frac{\partial c^{\hat{a}u}}{\partial \xi^m}\bigg|_{\xi_0}\frac{\partial c^{\hat{a}u}}{\partial \xi^n}\bigg|_{\xi_0}, \tag{8.19}$$

and the free energy density $\mathcal{F}$ is

$$\mathcal{F} = -T^4\left\{\Big[4+4h'_{11}+4(h'_{12}+1)\Big]G(0)+8\sum_i G\Big(\frac{M_i}{T}\Big)+\mathcal{O}\Big(e^{-\frac{M_{\text{min}}}{T}}\Big)\right\}. \tag{8.20}$$

The factor 8 in the above expression counts the number of boson/fermion pairs in the long vector multiplets of tree level mass M_i. The $\mathcal{O}\big(e^{-\frac{M_{\text{min}}}{T}}\big)$ term includes all contributions of the modes whose masses do not vanish in the neighborhood we are considering and thus admit a lower bound $M_{\text{min}} > M_i$. For $T < M_{\text{min}}$, these contributions are exponentially suppressed. Therefore $\mathcal{F}$ reaches its minimum when all M_i's vanish. Obviously this is true only at the conifold locus where $\xi^m = \xi^m_0$ where we have $c^{\hat{a}u} = 0$. Thus all flat directions ξ^m are lifted.

To obtain the scalar masses that $\mathcal{F}$ induces at one loop, we proceed as for the Coulomb branch. Therefore it seems that we need to find out the mass spectrum $\{M_i\}$ by expanding the tree-level scalar potential around the vacuum Eq.(8.17), to obtain an expression similar to Eq.(8.13). However this is very hard to achieve here since the mass-squared matrix is too complicated to diagonalize. Fortunately, observing Eq.(6.37) we see that actually only the sum of the mass squared $\sum_i M_i^2$ is evoked in the computation of one-loop mass. Thus we only need to simply compute the trace of the coefficient matrix, instead of computing the eigenvalues. The result is [19]:

$$\sum_i M_i^2 = 2e^{\mathcal{K}_{\text{V}}^{(0)}}g^{(0)\bar{\jmath}k}\,Q^{\hat{a}}_k Q^{\hat{a}}_j\,c^{\hat{a}u}c^{\hat{a}u}+\dots \tag{8.21}$$

Referring again to Eqs (6.36) and (6.37), the one-loop squared masses $M_m^{(\text{1-loop})}$ of the fields ξ^m at their minimum $\xi = \xi_0$ are determined by the matrix

$$\begin{aligned}\Lambda'^m{}_n &= \frac{1}{2}h^{(0)ml}\frac{\partial^2\mathcal{F}}{\partial\xi^l\partial\xi^n}\bigg|_{\xi_0} = \frac{T^2}{16}\,\frac{1}{2}h^{(0)ml}\,8\sum_i\frac{\partial^2 M_i^2}{\partial\xi^l\partial\xi^n}\bigg|_{\xi_0}\\ &= T^2e^{\mathcal{K}_{\text{V}}^{(0)}}g^{(0)\bar{\imath}j}\,Q^{\hat{a}}_j Q^{\hat{a}}_i\,h^{(0)ml}\frac{\partial c^{\hat{a}u}}{\partial\xi^l}\bigg|_{\xi_0}\frac{\partial c^{\hat{a}u}}{\partial\xi^n}\bigg|_{\xi_0},\end{aligned} \tag{8.22}$$

where we have used the fact that $c^{\hat{a}u}|_{\xi_0} = 0$ to obtain the last equality. The eigenvalues of Λ' are the desired squared masses we are looking for. Taking the trace, they satisfy

$$\sum_m\big(M_m^{(\text{1-loop})}\big)^2 = T^2e^{\mathcal{K}_{\text{V}}^{(0)}}g^{(0)\bar{\imath}j}\,Q^{\hat{a}}_j Q^{\hat{a}}_i\,h^{(0)nl}\frac{\partial c^{\hat{a}u}}{\partial\xi^l}\bigg|_{\xi_0}\frac{\partial c^{\hat{a}u}}{\partial\xi^n}\bigg|_{\xi_0}. \tag{8.23}$$

Thus, the ξ^m's have acquired a mass of order the temperature scale. Due to the arbitrariness in the choice of parametrization ξ^m of the D-term flat directions, all charged black hole hypermultiplets scalars $c^{\hat{a}u}$ have a mass of order T.

<table>
<tr><th rowspan="3"></th><th colspan="2">Scalars acquiring VEV's</th><th colspan="5">Superfields</th></tr>
<tr><th rowspan="2">In vector multiplets</th><th rowspan="2">In hyper-multiplets</th><th colspan="3">Vector multiplets</th><th colspan="2">Hypermultiplets</th></tr>
<tr><th>Massless (moduli)</th><th>Massive short</th><th>Massive long</th><th>Massless (moduli)</th><th>Massive</th></tr>
<tr><td>Coulomb phase</td><td>X^i</td><td>none</td><td>S</td><td>0</td><td>0</td><td>0</td><td>R</td></tr>
<tr><td>Higgs phase</td><td>none</td><td>$\mathscr{C}^{\hat{a}}$ mod. gauge orbits such that $\vec{D}_i = 0$</td><td>0</td><td>0</td><td>S</td><td>$R - S$</td><td>0</td></tr>
</table>

Table 8.1: Superfield contents in the Coulomb and Higgs branches (when $R > S$) associated to the $\mathcal{N} = 2$ $U(1)^S$ gauge theory coupled to R hypermultiplets, which is encountered in the neighborhood of a conifold locus in $\tilde{\mathcal{M}}_{\rm V} \times \tilde{\mathcal{M}}_{\rm H}$. The scalars X^p and q^λ of the massless spectator vector multiplets and hypermultiplets are not represented.

Cosmology and moduli stabilization

The cosmology induced by the effective actions (8.11) and (8.18) can be worked out following the lines in analyzing the model (6.31), provided that the free energy densities (8.12) and (8.20) fit into the form of Eq.(6.32). Therefore the resulting cosmology evolution follows the pattern of Eq.(6.34), which is a radiation-like universe where energy stored in the background scalar motions is proportional to that stored in radiation. Moduli involved in the conifold transition are attracted to the conifold locus while the spectator moduli X^p and q^λ are frozen at some value in the flat direction. Therefore cosmological evolution dynamically attracts the CY_3 to the singular configuration $\check{M}$.

8.2 Stabilization at a non-Abelian gauge symmetry locus

We consider moduli stabilization at the extremal transition locus described in Sec.4.3, where the CY_3 becomes singular due to the shrinking of 2-spheres along a rational complex curve instead of at isolated points. This configuration gives rise to $SU(N)$ gauge symmetry enhancement. Our aim is to show that the flat directions are lifted at one-loop level, and the internal CY_3 is attracted to the singular configuration.

Non-Abelian gauge theory at extremal transition locus

Keeping to the setup in Sec.4.3, we let the initial non-singular CY_3 be denoted by M and the singular configuration arising from shrinking 2-spheres be $\check{M}$. We let the shrinking 2-spheres represent $N-1$ homologically independent 2-cycles with intersection matrix of type A_{N-1}. Suppose the resulting singular curve in $\check{M}$ has genus g. Then there are $(g-1)(N^2-N)$ non-toric deformations for desingularizing $\check{M}$ into another manifold M'', where the change in Hodge number is given in Eq.(4.11). The branch of M'' exists only when $g \geq 2$.

All analysis in the rest of this section is carried out in the cases with $g \geq 2$. Actually, when $g = 0$, the pure $SU(N)$ gauge theory is asymptotically free and is Abelian in the IR, with gauge group $U(1)^{N-1}$. Thus, this situation is expected to be dual to a particular example of the conifold case we have already studied, for $R = S = N-1$. For $g = 1$, the vector and hypermultiplet in the adjoint representation combine into an $\mathcal{N} = 4$ $SU(N)$ gauge sector. This case is conformal and has already been considered in Chapter 7, leading to an attraction of the moduli at the origin of the Coulomb branch, thus restoring the full non-Abelian symmetry. On the contrary, new physics is encountered for $g \geq 2$, since the $SU(N)$ gauge theory is non-asymptotically free and moreover admits Coulomb and Higgs branches.

We write down the low energy effective action in M near the singular locus. The $\mathcal{N}_4 = 2$ supermultiplets representing perturbative modes include the h_{11} massless vector multiplets and the $h_{12}+1$ massless hypermultiplets. There are also N^2-N vector multiplets arising from D2-brane wrapping shrinking 2-spheres, which combine with $N-1$ of the perturbative vector multiplet to give the $SU(N)$ gauge group. Moreover there are $g(N^2-N)$ extra non-perturbative hypermultiplets, which combine with $g(N-1)$ of the perturbative ones giving rise to g hypermultiplets transforming in the adjoint of $SU(N)$. We let $\tilde{\mathcal{M}}_{\mathrm{V}}$ and $\tilde{\mathcal{M}}_{\mathrm{H}}$ be the space of vector multiplet scalars and the space of hypermultiplet scalars. The effective action is just as Eq.(8.1), with X^I ($I = 1, \ldots, h_{11}+N^2+N$) complex coordinates of $\tilde{\mathcal{M}}_{\mathrm{V}}$, and q^Λ ($\Lambda = 1, \ldots, 4h_{12}+4g(N^2-N)+4$) real coordinates on $\tilde{\mathcal{M}}_{\mathrm{H}}$.

$$S_{\text{tree}} = \int d^4x \sqrt{-g}\left\{\frac{\mathcal{R}}{2} - g_{I\bar{J}}\nabla X^I \nabla \bar{X}^J - h_{\Lambda\Sigma}\nabla q^\Lambda \nabla q^\Sigma - \mathcal{V}\right\}, \tag{8.24}$$

Here all quantities appearing in the action are non-singular at the locus of $\check{M}$, since non-perturbative D2-brane states are "integrated in". The covariant derivatives are with respect to the $SU(N)$ gauge symmetry. We further constrain the action using conditions from the physics that we expect. The latter require that $\tilde{\mathcal{M}}_{\mathrm{V}}$ contains a (N^2-1)-dimensional (complex dimension) subspace of $SU(N)$-isometry, and $\tilde{\mathcal{M}}_{\mathrm{H}}$ has a $g(N^2-1)$ dimensional (quaternionic dimension) subspace of $SU(N)$-isomety. It can be shown that the coordinates of $\tilde{\mathcal{M}}_{\mathrm{V}}$ can be chosen in the following way

$$\{X^I\} = \{X^a; X^p\}, \quad (a = 1, \ldots, N^2-1;\ p = N^2, \ldots, h_{11}+N^2-N), \tag{8.25}$$

Here a is the gauge index of $SU(N)$, transforming in the adjoint; index p is for labeling fields neutral under $SU(N)$, which we call spectator fields. On the other hand, the quaternionic manifold $\tilde{\mathcal{M}}_{\rm H}$ accommodates the following coordinates

$$\begin{aligned}&\{q^\Lambda\} = \{c^{a\mathcal{A}u}; q^\lambda\}\,,\\&\big(\mathcal{A} = 1,\ldots,g;\; u = 1,2,3,4;\; \lambda = 4g(N^2-1)+1,\ldots,4h_{12}+4g(N^2-1)+4\big)\,,\end{aligned}\tag{8.26}$$

where we have the same separation of spectator fields, neutral under $SU(N)$ labeled by λ, from those transforming in the adjoint of $SU(N)$ carrying the gauge index a. Moreover each hypermultiplet transforming in the adjoint of $SU(N)$ is labeled by $\mathcal{A}$, containing 4 real components labeled by u. The action of the triplet of complex structures only touches the index u. With the above choice of coordinate system, the singular configuration $\check{M}$ arises at the locus $X^a = 0 = c^{a\mathcal{A}u}$.

Using the formulae of $\mathcal{N}_4 = 2$ supergravity, we can work out more details of the action (8.24). The strategy is the same as for the conifold case: we expand all quantities in terms of X^a and $c^{a\mathcal{A}u}$ the fields vanishing at the singular locus. We find that the kinetic terms become

$$g_{I\bar{J}}\nabla X^I\nabla\bar{X}^J = l^2\nabla X^a\nabla\bar{X}^a + g^{(0)}_{p\bar{q}}\partial X^p\partial\bar{X}^q + \ldots\,,\tag{8.27}$$

$$h_{\Lambda\Sigma}\nabla q^\Lambda\nabla q^\Sigma = \frac{1}{2}\nabla c^{a\mathcal{A}u}\nabla c^{a\mathcal{A}u} + h^{(0)}_{\lambda\sigma}\partial q^\lambda\partial q^\sigma + \ldots\,,\tag{8.28}$$

where only the leading contributions are displayed, and the superscript "(0)" symbolizes the leading order in the expansion. The $SU(N)$-isometric subspace has diagonalized metric in the leading order. It can be realized by proper parameterization that $g^{(0)}_{ab} = l^2\delta_{ab}$ for vector multiplets where l^2 depends spectators X^p, and also $h^{(0)}_{a\mathcal{A}u,b\mathcal{B}v} = \delta_{ab}\delta_{\mathcal{A}\mathcal{B}}\delta_{uv}$ for hypermultiplets. We can also obtain the leading contribution to the scalar potential, but we are not intended to show all the details here, which are just applications of standard supergravity formulae [19]. Finally we arrive at the following form of effective action, which is valid locally close to the non-Abelian locus:

$$\begin{aligned}S_{\rm tree} = \int d^4x\,\sqrt{-g}\bigg\{&\frac{\mathcal{R}}{2} - l^2\,\nabla_\mu X^a\nabla^\mu\bar{X}^a - g^{(0)}_{p\bar{q}}\,\partial_\mu X^p\partial^\mu\bar{X}^q - \frac{1}{2}\,\nabla_\mu c^{a\mathcal{A}u}\nabla^\mu c^{a\mathcal{A}u} - h^{(0)}_{\lambda\sigma}\,\partial q^\lambda\partial q^\sigma\\&- e^{\mathcal{K}^{(0)}_{\rm V}}\Big(l^2[X,\bar{X}]^a[X,\bar{X}]^a + 2[X,c^{\mathcal{A}u}]^a[c^{\mathcal{A}u},\bar{X}]^a + \frac{1}{4l^2}\vec{D}^a\cdot\vec{D}^a\Big) + \cdots\bigg\}.\end{aligned}\tag{8.29}$$

We have introduced the D-terms

$$\vec{D}^a := -i\,f^{abc}\,\mathscr{C}^{b\mathcal{A}\dagger}\vec{\sigma}\mathscr{C}^{c\mathcal{A}},\quad\text{with}\quad \mathscr{C}^{a\mathcal{A}} = \begin{pmatrix} i(c^{a\mathcal{A}1}+ic^{a\mathcal{A}2})\\ (c^{a\mathcal{A}3}+ic^{a\mathcal{A}4})^*\end{pmatrix}\ \text{the } SU(2)_{\mathcal{R}}\text{ doublets,}\tag{8.30}$$

where f^{abc} are structure constants of the $SU(N)$ group. Therefore, the Lagrangian density in Eq.(8.29), up to the "spectator multiplets", has the form of a minimally coupled rigid $\mathcal{N} = 2$

supersymmetric $SU(N)$ super Yang-Mills theory coupled to g hypermultiplets in the adjoint representation, formally coupled to gravity. Moreover no surprisingly, the scalar potential is the same as in the field theory discussion Eq.(4.13).

The flat directions defined by the potential contains the singular locus corresponding to $\check{M}$. At a generic vacuum in these flat directions, masses are generated through Higgs mechanism for the degrees of freedom involved in the $SU(N)$ gauge theory. These masses vanish exactly at the locus of $SU(N)$-enhanced symmetry, and can lift the tree-level flat directions at one-loop level. Actually, in carrying out this computation at one-loop, we will rather need the sum of all these mass squared than the masses themselves, just as what we computed in Eq.(8.21) in the Higgs branch of conifold transition. This mass squared sum can be worked out explicitly with the action (8.29). We omit the detailed computation, which is explained in [19]. The total mass squared turns out to be

$$\sum M^2 = 16Ne^{\mathcal{K}_{\rm V}^{(0)}}\left[(g+1)X^a\bar{X}^a + \frac{1}{l^2}c^{a\mathcal{A}u}c^{a\mathcal{A}u}\right] + \dots, \tag{8.31}$$

where X^a and $c^{a\mathcal{A}u}$ are set to be in the flat directions of the scalar potential in (8.29).

Lifting flat directions at finite temperature

At finite temperature in weak coupling regime, the tree level action (8.29) is corrected by Colemann-Weiberg effective potential. We now compute this the one-loop correction and the result has to include explicitly the whole set of light degrees of freedom, including the $SU(N)$ gauge sector coupled to g hypermultiplets in the adjoint. Also, since the space of tree level flat directions around the $SU(N)$ enhanced symmetry locus splits into the Coulomb branch and the Higgs branch, each giving rise to different light degrees of freedom, the computation should be carried out separately in these two branches.

- **Coulomb branch**

The Coulomb phase corresponds to scalar VEV's such that the matrices X^aT^a and $c^{a\mathcal{A}u}T^a$ sit in the Cartan sub-algebra, where T^a are generators of $SU(N)$. It is not hard to see that three terms in the scalar potential in Eq.(8.29) vanish separately. Denoting as T^i ($i = 1,\dots,N-1$) the Cartan generators of $SU(N)$ and T^m ($m = N,\dots,N^2-N$) the remaining ones, we have

$$\text{Coulomb branch:}\left\{\left(X^i \text{ arbitrary}, X^m = 0, c^{i\mathcal{A}u} \text{ arbitrary}, c^{m\mathcal{A}u} = 0\right)\right\}\times\left\{\left(X^p, q^\lambda\right) \text{ arbitrary}\right\}, \tag{8.32}$$

which corresponds to a compactification on the space M. There is a subtlety following different ways of assigning VEV's to X^i and $c^{a\mathcal{A}u}$. When we sit in the the vacuum with X^i's generic but $c^{i\mathcal{A}u}=0$, the non-Cartan vector multiplets and hypermultiplets acquire masses. From the former we obtain N^2-N short massive vector multiplets, and from the latter, we have $g(N^2-N)$ massive hypermultiplets. If we assign nonzero VEV also to $c^{i\mathcal{A}u}$, then the N^2-N short massive vector multiplets absorb N^2-N of the massive hypermultiplets, forming N^2-N long vector multiplets. The complete superfield content in this case is reported in Table 8.2.

Restricted to the weak string coupling regime, we are now ready to write down the one-loop thermal effective action. In the Coulomb branch, it amounts to adding the tree level action (8.29) in some vacuum (8.32) to the one-loop Coleman-Weinberg effective potential $\mathcal{F}$,

$$S_{\text{1-loop}}=\int d^4x\sqrt{-g}\left\{\frac{\mathcal{R}}{2}-l^2\partial X^i\partial\bar{X}^j-g^{(0)}_{p\bar{q}}\partial X^p\partial\bar{X}^q-\frac{1}{2}\partial c^{a\mathcal{A}u}\partial c^{a\mathcal{A}u}-h^{(0)}_{\lambda\sigma}\partial q^\lambda\partial q^\sigma-\mathcal{F}\right\}. \tag{8.33}$$

Following the logic in obtaining Eq.(5.52), $\mathcal{F}$ in the present case is

$$\mathcal{F}=-T^4\left\{\Big[4+4h_{11}+4(h_{12}+1)\Big]G(0)+\sum_{\hat{s}}G\Big(\frac{M_{\hat{s}}}{T}\Big)+\mathcal{O}\big(e^{-\frac{M_{\min}}{T}}\big)\right\}, \tag{8.34}$$

where the index $\hat{s}$ labels all pairs of degenerate boson-fermion states in the massive vector multiplets and hypermultiplets involved in the $SU(N)$ gauge theory and collected in Table 8.2. In Eq.(8.34), we take the temperature to be below the lower bound $M_{\min}>0$, in the vicinity of the $SU(N)$-symmetry locus, of the remaining masses of the full string spectrum. $\mathcal{F}$ is minimized when all classical masses in the $SU(N)$ gauge sector vanish, $\forall\hat{s}:\ M_{\hat{s}}=0$. Using the general formula (8.31) in the Coulomb branch,

$$\sum_{\hat{s}}M^2_{\hat{s}}=16Ne^{\mathcal{K}^{(0)}_{\text{V}}}\Big[(g+1)X^i\bar{X}^i+\frac{1}{l^2}\,c^{i\mathcal{A}u}c^{i\mathcal{A}u}\Big]+\cdots, \tag{8.35}$$

this implies $X^i=0$, $c^{i\mathcal{A}u}=0$. Therefore, all moduli involved in the Coulomb phase of the $SU(N)$ gauge theory are lifted. The kinetic terms of X^i and $c^{i\mathcal{A}u}$ being diagonal, it is straightforward to compute the scalar masses induced by $\mathcal{F}$ using Eqs (6.36) and (6.37):

$$\big(M^{(\text{1-loop})}_i\big)^2=\frac{1}{l^2}\,\frac{\partial^2\mathcal{F}}{\partial X^i\partial\bar{X}^i}\bigg|_{X^j=c^{j\mathcal{A}u}=0}=\frac{T^2}{16}\,\frac{1}{l^2}\sum_s\frac{\partial^2M^2_s}{\partial X^i\partial\bar{X}^i}\bigg|_{X^j=c^{j\mathcal{A}u}=0}=T^2\,(g+1)\,\frac{N}{l^2}e^{\mathcal{K}^{(0)}_{\text{V}}}, \tag{8.36}$$

$$\big(M^{(\text{1-loop})}_{i\mathcal{A}u}\big)^2=\frac{\partial^2\mathcal{F}}{\partial c^{i\mathcal{A}u}\partial c^{i\mathcal{A}u}}\bigg|_{X^j=c^{j\mathcal{A}u}=0}=\frac{T^2}{16}\sum_s\frac{\partial^2M^2_s}{\partial c^{i\mathcal{A}u}\partial c^{i\mathcal{A}u}}\bigg|_{X^j=c^{j\mathcal{A}u}=0}=T^2\,2\,\frac{N}{l^2}e^{\mathcal{K}^{(0)}_{\text{V}}}. \tag{8.37}$$

Due to the arbitrariness in the choice of Cartan subalgebra at the origin of the Coulomb branch, we conclude that all vector multiplet and hypermultiplet scalars X^a and $c^{a\mathcal{A}u}$, even though classically massless, have one-loop masses given by Eqs (8.36) and (8.37) respectively.

	Scalars acquiring VEV's		Superfields				
	In vector multiplets	In hyper-multiplets	Vector multiplets			Hypermultiplets	
			Massless (moduli)	Massive short	Massive long	Massless (moduli)	Massive
Coulomb phase	X^i	none	$N-1$	N^2-N	0	$g(N-1)$	$g(N^2-N)$
	X^i or none	$c^{i\mathcal{A}u}$	$N-1$	0	N^2-N	$g(N-1)$	$(g-1)(N^2-N)$
Higgs phase	none	$\mathscr{C}^{a\mathcal{A}}$ mod. gauge orbits such that $\vec{D}^a=0$	0	0	N^2-1	$(g-1)(N^2-1)$	0

Table 8.2: Superfield contents in the Coulomb and Higgs branches (when $g \geq 2$) associated to the $\mathcal{N}=2$ $SU(N)$ gauge theory coupled to g hypermultiplets in the adjoint representation, which is encountered in the neighborhood of a non-Abelian locus in $\tilde{\mathcal{M}}_{\rm V} \times \tilde{\mathcal{M}}_{\rm H}$. The scalars X^p and q^α of the massless spectator vector multiplets and hypermultiplets are not represented. At special loci in the Coulomb branch, where some $X^i = X^j$ and $c^{i\mathcal{A}u} = c^{j\mathcal{A}u}$ for $i \neq j$, some generically massive multiplets are actually massless, and the $SU(N)$ gauge symmetry is broken to a non-Abelian subgroup of rank $N-1$, rather than $U(1)^{N-1}$.

- **Higgs branch**

The Higgs branch vacua are defined by the D-term constraints $\vec{D}^a = 0$. We have

$$\text{Higgs branch}: \quad \Big\{ \big(X^a = 0, \mathscr{C}^{a\mathcal{A}} \text{ such that } \vec{D}^b = 0 \big) \Big\} \times \Big\{ \big(X^p, q^\alpha \big) \text{ arbitrary} \Big\}. \tag{8.38}$$

The above conditions fix $3(N^2-1)$ components among the $4g(N^2-1)$ scalars $c^{a\mathcal{A}u}$. Among the flat directions that rest, another N^2-1 are $SU(N)$ gauge orbit. Fixing these gauge degrees of freedom amounts to gauging away N^2-1 would-be-Goldstone bosons. Therefore in the end $4(g-1)(N^2-1)$ flat directions of inequivalent vacua remain. By supersymmetry, the latter can be parameterized by the scalars of $(g-1)(N^2-1)$ neutral hypermultiplets. Thus the Higgs branch exists only for $g \geq 2$, in which case it is realized geometrically by compactifying on M'' with Hodge numbers given in Eq.(4.11). Actually, N^2-1 of the initial $g(N^2-1)$ hypermultiplets combine with the Higgsed vector multiplets into N^2-1 massive long vector multiplets, as summarized in Table 8.2.

We parameterize the Higgs branch flat directions using a set of coordinates ξ^m $\big(m = 1, \ldots, 4(g-1)(N^2-1)\big)$. They satisfy $f^{abc}\mathscr{C}^{b\mathcal{A}\dagger}(\xi)\sigma^x\mathscr{C}^{c\mathcal{A}}(\xi) = 0$ and the Jacobian matrix $\left(\frac{\partial c^{a\mathcal{A}u}}{\partial \xi^m}\right)$ is of rank $4(g-1)(N^2-1)$. The origin of the Higgs branch is denoted ξ_0^m. In these notations, the one-loop effective action of the type IIA string theory compactified on M'' at finite temperature is

$$S_{\text{1-loop}} = \int d^4x\, \sqrt{-g} \left\{ \frac{\mathcal{R}}{2} - g^{(0)}_{p\bar{q}}\, \partial X^p \partial \bar{X}^q - h^{(0)}_{mn}\, \partial \xi^m \partial \xi^n - h^{(0)}_{\lambda\sigma}\, \partial q^\lambda \partial q^\sigma - \mathcal{F} \right\}, \tag{8.39}$$

where the induced metric of the ξ^m's is

$$h^{(0)}_{mn} = \frac{1}{2} \frac{\partial c^{a\mathcal{A}u}}{\partial \xi^m}\bigg|_{\xi_0} \frac{\partial c^{a\mathcal{A}u}}{\partial \xi^n}\bigg|_{\xi_0}. \tag{8.40}$$

Mimicking once more Eq.(5.52), we write down the free energy density

$$\mathcal{F} = -T^4 \left\{ \Big[4 + 4h''_{11} + 4(h''_{12} + 1) \Big] G(0) + 8 \sum_a G\Big(\frac{M_a}{T}\Big) + \mathcal{O}\big(e^{-\frac{M_{\text{min}}}{T}}\big) \right\}, \tag{8.41}$$

where the factor 8 counts the number of boson-fermion pairs of states in the long vector multiplets of masses M_a ($a = 1, \ldots, N^2 - 1$). The contributions of all the other massive modes of the spectrum are exponentially suppressed, when $T < M_{\text{min}}$.

Obviously $\mathcal{F}$ is minimized locally when all masses M_a vanish. Applying Eq.(8.31), we have the sum of all the mass squared

$$8 \sum_a M_a^2 = 16 \frac{N}{l^2} e^{\mathcal{K}^{(0)}_{\text{V}}}\, c^{a\mathcal{A}u} c^{a\mathcal{A}u} + \cdots, \tag{8.42}$$

showing that the vanishing of all M_a is achieved only at the origin of the Higgs branch $c^{a\mathcal{A}u} = 0$, or $\xi^m = \xi^m_0$. Therefore all classical flat directions ξ^m are lifted. The squared mass matrix of the ξ^m's induced by $\mathcal{F}$ is

$$\Lambda''^m{}_n = \frac{1}{2} h^{(0)ml} \frac{\partial^2 \mathcal{F}}{\partial \xi^l \partial \xi^n}\bigg|_{\xi_0} = \frac{T^2}{16} \frac{1}{2} h^{(0)ml}\, 8 \sum_a \frac{\partial^2 M_a^2}{\partial \xi^l \partial \xi^n}\bigg|_{\xi_0} = T^2\, 2 \frac{N}{l^2} e^{\mathcal{K}^{(0)}_{\text{V}}} \delta^m_n, \tag{8.43}$$

where we have used the fact $c^{a\mathcal{A}u}|_{\xi_0} = 0$ to reach the last equality. Thus, the ξ^m's are mass eigenstates and degenerate. Since the parametrization of the Higgs branch was chosen arbitrarily, we obtain that all scalars $c^{a\mathcal{A}u}$ acquire a common mass given by Eq.(8.43). Consistently, this is the result we already found by approaching the $SU(N)$ non-Abelian locus from the Coulomb branch, Eq.(8.37).

The cosmology induced by the above effective actions (8.29) and (8.39) is the same as in the case of conifold transition discussed in the end of Sec.8.1. We end up with a radiation-like universe evolving in the pattern of Eq.(6.34), with moduli involved in the extremal transition stabilized and spectator moduli frozen in the flat directions. The internal CY_3 is dynamically attracted to the singular configuration $\check{M}$.

8.3 Intersections of extremal transition loci

The previous sections have shown that with cosmological evolution, type II strings tend to settle the vacuum at singular loci of the internal CY_3. Therefore when the moduli space allows several extremal transition loci, naturally the intersection points of these loci are the preferred choices of vacuum. In this section we illustrate this fact with an example where the internal CY_3 can develop on the same time isolated singular nodes and a rational curve of uniform singularity. When the type IIA(IIB) is attracted to the vacuum corresponding to this singular configuration of the internal CY_3, we will have the Kähler (complex structure) moduli space is completely lifted, together with some of the complex structure (Kähler) moduli. This implies in particular that the axio-dilaton field of the heterotic dual description is stabilized.

Example: intersection of a conifold locus and a non-Abelian locus

We consider the example in [63, 64] of small Hodge number h_{11}. The type IIA model we analyze at finite temperature is compactified on a CY manifold M obtained by resolving the orbifold singularities of a degree 12 hypersurface in $\mathbb{P}^4_{(1,1,2,2,6)}$. Denoting the projective coordinates as $x_1, \dots, x_5$, the ambient space presents initially a singularity of type A_1 at $x_1 = x_2 = 0$. This singular locus gives rise to a genus 2 singular curve restricted to the degree 12 hypersurface. Blowing up the ambient space singularity we obtain the non singular Calabi-Yau space M of Hodge numbers $(h_{11}, h_{12}) = (2, 128)$. Obviously, the Kähler moduli space $\mathcal{M}_{\rm V}$ admits a non-Abelian locus with $N = 2$ and $g = 2$. Moreover we also observe a conifold locus of M, which can be revealed in the mirror manifold.

In mirror IIB picture, the model is defined by the vanishing locus of degree 12 polynomials in $\mathbb{P}^4_{(1,1,2,2,6)}$, modded out by a $\mathbb{Z}_6^2 \times \mathbb{Z}_2$ group. We denote this mirror CY_3 by W. The most general hypersurface consistent with this action is [46, 48, 63, 64]

$$\mathscr{P} = x_1^{12} + x_2^{12} + x_3^6 + x_4^6 + x_5^2 - 12\psi\, x_1 x_2 x_3 x_4 x_5 - 2\phi\, x_1^6 x_2^6\,, \tag{8.44}$$

which admits 2 complex structure deformations, ψ and ϕ. Thus, the mirror Calabi-Yau space W admits $h_{11} = 2$ Kähler moduli. Defining

$$z_1 = -\frac{1}{864}\frac{\phi}{\psi^6}\,, \qquad z_2 = \frac{1}{\phi^2}\,, \tag{8.45}$$

some simple algebra yields the degeneracy loci of the hypersurface $\mathscr{P} = 0$ which are defined by the vanishing of $\Delta_c \Delta_{nA}$, where

$$\Delta_c \equiv (1 - z_1)^2 - z_1^2 z_2\,, \qquad \Delta_{nA} \equiv 1 - z_2\,. \tag{8.46}$$

When $\Delta_c = 0$, isolated conical singularities arise, but are all identified by the orbifold action of $\mathbb{Z}_6^2 \times \mathbb{Z}_2$. Thus it is the conifold locus, with $R = S = 1$. Therefore the conifold does not allow desingularization by deformation, so the resulting effective gauge theory has no Higgs branch. The locus $\Delta_{nA} = 0$ is just the non-Abelian locus with $g = 2$ and $N = 2$, where the enhanced gauge group is $SU(2)$ [63,64]. At this singular configuration, $(g-1)(N^2 - N) = 2$ non toric deformations are available, leading to a distinct smooth Calabi-Yau space M''. The ambient spaces, degrees of polynomials and Hodge numbers of the families of CY manifolds on either side of the associated non-Abelian extremal transition are [46, 48]

$$\mathbb{P}^4_{(1,1,2,2,6)}[12](2,128) \longleftrightarrow \mathbb{P}^5_{(1,1,1,1,1,3)}[2,6](1,129). \tag{8.47}$$

The conifold and non-Abelian loci intersect at two points on the compactified moduli space $\mathcal{M}_V$ [63],

$$(z_1, z_2) = (1/2, 1) \quad \text{or} \quad (\infty, 1). \tag{8.48}$$

We are interested in the effective gauge theory near these intersection points. We carry out the analysis in the branch M, which is the Coulomb branch of both the non-Abelian locus and the conifold locus. We first identify $h_{11} = 2$ perturbative vector multiplets and $h_{12}+1 = 129$ perturbative hypermultiplets. The gauge group is $U(1)_{\text{grav}} \times U(1)_{\text{con}} \times U(1)_{\text{nA}}$, where the first factor is due to the graviphoton, the second arises from the shrinking 2-cycle at conifold point, and the third is induced by the shrinking 2-cycle at the $SU(2)$ enhanced symmetry point, supplying the Cartan component of $SU(2)$. Then in addition to the perturbative field contents, we have one blackhole hypermultiplet charged under $U(1)_{\text{con.}}$, and also all the non Cartan components responsible for the enhancement of $U(1)_{\text{nA}} \to SU(2)$, including 2 non-perturbative vector multiplets and 4 non-perturbative hypermultiplets. We denote the spaces of all these light vector multiplet scalars and hypermultiplet scalars by $\tilde{\mathcal{M}}_{\text{V}}$ and $\tilde{\mathcal{M}}_{\text{H}}$ respectively.

Going through the same procedure as in the previous two sections, we can establish the low energy effective action locally about the intersections (8.48). The coordinates of $\tilde{\mathcal{M}}_{\text{V}}$ and $\tilde{\mathcal{M}}_{\text{H}}$ can be chosen respectively as

$$\begin{aligned} &\{X^1; X^a\} \ \text{ and } \ \{c^{1u}; c^{a\mathcal{A}u}; q^\lambda\}, \\ &a = 2,3,4;\ u = 1,2,3,4;\ \mathcal{A} = 1,2;\ \lambda = 4\times 7+1, \dots, 4\times 129 + 4\times(1+4)\,. \end{aligned} \tag{8.49}$$

Here X^1 is the scalar partner of the $U(1)_{\text{con}}$ gauge boson and X^a is in the adjoint of $SU(2)$. Similarly, c^{1u} are the components of the black hole hypermultiplet, while $c^{a\mathcal{A}u}$ are those of the two hypermultiplets in the adjoint of $SU(2)$. The scalars of the 127 hypermultiplets that are neutral with respect to $U(1)_{\text{con}} \times SU(2)$ are denoted q^λ. The conifold locus is given by $X^1 = 0 = c^{1u}$, and the non-Abelian locus is given by $X^a = 0 = c^{a\mathcal{A}u}$. We call the intersection of the tow loci P_0. The local effective action around P_0 takes the form

$$S_{\text{tree}} = \int d^4x\,\sqrt{-g}\,\Big\{\frac{\mathcal{R}}{2} - l_1^2\,\partial X^1\partial\bar{X}^1 - l^2\,\nabla X^a\nabla\bar{X}^a - \frac{1}{2}\,\nabla c^{1u}\nabla c^{1u} - \frac{1}{2}\,\nabla c^{a\mathcal{A}u}\nabla c^{a\mathcal{A}u} - h^{(0)}_{\lambda\sigma}\,\partial q^\lambda\partial q^\sigma$$
$$- e^{\mathcal{K}^{(0)}_{\text{V}}}\Big(2\,|X^1|^2\,c^{1u}c^{1u} + \frac{1}{4l_1^2}\,(c^{1u}c^{1u})^2\Big)$$
$$- e^{\mathcal{K}^{(0)}_{\text{V}}}\Big(l^2[X,\bar{X}]^a[X,\bar{X}]^a + 2[X,c^{\mathcal{A}u}]^a[c^{\mathcal{A}u},\bar{X}]^a + \frac{1}{4l^2}\vec{D}^a\cdot\vec{D}^a\Big) + \cdots\Big\}, \qquad (8.50)$$

where we have expanded all the quantities in terms of the vanishing scalar fields and keep only the leading order. We have l_1^2, l^2 the non-vanishing leading-order components of the Kähler metric on $\tilde{\mathcal{M}}_{\text{V}}$ at P_0, and $h^{(0)}_{\lambda\sigma}$ the leading-order metric at P_0 of the subspace spanned by neutral hypermultiplet scalars $\{q^\lambda\}$ in $\tilde{\mathcal{M}}_{\text{H}}$. The D-terms $\vec{D}^a$ are just as defined in Eq.(8.30).

We notice in the action (8.50) that the part concerning the conifold locus and the part concerning the non-Abelian locus are completely decoupled, so the analysis of moduli stabilization in Sec.8.1 and Sec.8.2 can be applied to the two parts separately. Since the conifold locus does not lead to a Higgs branch, we examine the Coulomb and the Higgs branch of the $SU(2)$-singular locus, corresponding to the internal spaces of M and M''.

The one-loop corrections attracts the scalars X^1, X^a, c^{1u} and $c^{a\mathcal{A}u}$ at zero and these scalars acquire masses of order the temperature scale, while the q^λ's remain flat directions of the thermal effective potential. Moreover, the full $U(1)_{\text{grav}} \times U(1)_{\text{con}} \times SU(2)$ gauge theory is restored. In the Coulomb branch corresponding to compactification on M, we have:

• The $h_{11} = 2$ Kähler moduli are stabilized in one of the two minima given in Eq.(8.48). Note that these two Kähler moduli are some reparameterization of z_1 and z_2 in Eq.(8.46).

• The scalars of $g(N-1) = 2$ hypermultiplets are stabilized at the origin of the Coulomb branch of $SU(2)$ in $\mathcal{M}_{\text{H}}$.

• The scalars of the $h_{12} + 1 - g(N-1) = 127$ left-over hypermultiplets remain flat directions in $\mathcal{M}_{\text{H}}$.

Similarly, sitting in the Higgs branch, corresponding to compactification on M'', we have:

• The $h''_{11} = 1$ complexified Kähler modulus parameterizing $\mathcal{M}''_{\text{V}}$ is stabilized.

• The scalars of $(g-1)(N^2-1) = 3$ hypermultiplets are stabilized at the origin of the Higgs branch of $SU(2)$ in $\mathcal{M}''_{\text{H}}$.

• The scalars of the $h''_{12}+1-(g-1)(N^2-1) = 127$ left-over hypermultiplets remain flat directions in $\mathcal{M}''_{\text{H}}$.

The heterotic dual

At zero temperature the type IIA string compactified on M has a heterotic dual compactified on $K3\times T^2$, due to the fact that the family $\mathbb{P}^4_{(1,1,2,2,6)}[12]$ are $K3$-fibrations [65]. The dual heterotic model is constructed in [66], which contains the axio-dilaton modulus S_h and the modulus T_h of the torus as vector multiplet moduli. The other torus modulus U_h is identified with T_h: in fact $T_h - U_h$ is projected out. The heterotic model also has 129 hypermultiplet moduli. The duality map for the vector multiplet moduli can be more precisely specified in the large complex volume limit, $S_h \to +\infty$, under which one has

$$z_1 = \frac{1728}{j(T_h)} + \cdots, \qquad z_2 = e^{-S_h} + \cdots, \tag{8.51}$$

where j is the $SL(2,\mathbb{Z})$-invariant modular form. Therefore, $z_2 \to 0$ and the two roots of the discriminant locus Δ_c in Eq.(8.46) merge into $z_1 = 1$. Actually, the heterotic string develops an $SU(2)$ enhanced gauge symmetry when $T_h = i$ modulo the classical T-duality group $SL(2,\mathbb{Z})$, in perfect agreement with Eq.(8.51) for $z_1 = 1$. This $SU(2)$-enhancement should not be confused with the $SU(2)$ gauge group occurring at the type II non-Abelian locus. Moreover, when S_h is finite, the conifold locus splits into two branches, as predicted by the exact pure $SU(2)$ $\mathcal{N} = 2$ super-Yang-Mills theory [41]. Being asymptotically free, the latter reduces in the IR to a $U(1)$ gauge theory coupled to a single (dyonic) hypermultiplet, realized as $U(1)_{\text{con}}$ in the type II setup [67,68].

This duality stay valid when we switch on finite temperature on both theories. This follows from the adiabatic argument in [59] mentioned in the beginning of Chapter 7. Therefore, the stabilization of the complex structure moduli z_1, z_2 of W at one of the two points in Eq.(8.48) translates immediately into a stabilization of the torus modulus T_h and axio-dilaton S_h in the dual heterotic model at finite temperature. As seen in Eq.(8.51), the obtained value of S_h corresponds to a strong coupling regime of the heterotic theory.

Discussion

In the specific example we have shown the possibility of lifting the whole Kähler moduli space, and the stabilization of the axio-dilaton modulus in the dual heterotic string theory. However this situation is not a luxury arising only in delicately coined-up models, but is general. Let us stay in the type IIA picture. On general grounds, one can make all $(1,1)$-type 2-cycles in a CY_3 shrink to zero size by sitting at the tip of the Kähler cone. Since shrinking 2-spheres imply extra massless D2-brane states, locally the locus where all $(1,1)$-type 2-cylces vanish gives rise to a maximum number of massless states. Therefore the tip of the Kähler cone can attract all Kähler moduli. It follows that the whole vector multiplet moduli space is lifted in the dual heterotic theory compactified on $K3 \times T^2$. Therefore the flat direction of the heterotic dilaton can be lifted since it lives in a vector multiplet. Mirror symmetry implies that there exists also a locus where all $(1,2)$-type 3-cycles in the CY_3 shrink to zero size. However this locus attracts only part of the complex structure moduli with the mechanism considered in this chapter because the latter infers extra massless states only for the cases where shrinking 3-spheres can be resolved to 2-spheres. Finally we are left with the moduli in the universal hypermultiplet. They cannot be stabilized by the mechanism considered in this chapter, since they are not associated to the geometrical deformations of the CY_3. Maybe the stabilization of the hypermultiplet moduli can be inferred from the dual heterotic side, since the hypermultiplet moduli space is exact for heterotic string compactified on $K3 \times T^2$.

Chapter 9

Conclusion

In this thesis, we depicted systematically string theory approach to cosmology at finite temperature, from the underlying principles, the theoretical setup until the applications. To conclude, we summarize the essential points below.

The thermal/quantum induced cosmology

After introducing necessary string theory elements, we have set up the thermal string scenario, which has the goal of describing cosmology in a single unified theoretical framework. The basic assertion is to let the cosmological evolution be dictated by the effective supergravity of the string theory. Thus all aspects in cosmological evolution: spacetime metric evolution, matter contents, interactions are all derived from first principle, and unified in one single quantum description. This overcomes the drawback of ΛCDM model that matter contents are postulated and that gravity is not quantized.

The tree level effective supergravity only allows trivial static solutions with flat or AdS background. It is only after taking into account thermal/quantum corrections beyond tree level that the universe sets out to evolve nontrivially. Given that the cosmological constant is small, we choose the no-scale type supergravity at tree level which have vanishing cosmological constant. We restrict ourselves to weakly coupled regime, and compute the thermal/quantum corrections perturbatively up to one-loop level. In fact this one-loop correction is just the string partition function computed against a thermal background, and it induces a Colemann-Weinberg effective potential pending to the tree-level effective action. It then turns out that the corrected effective action describes a universe filled with an ideal string gas at finite temperature, and the one-loop effective potential is nothing but the Helmholtz free energy density. An overview of thermal string cosmological evolution is illustrated in the thesis, which we summarize here in the chronological order.

The thermal one-loop amplitude in string theories meets with Hagedorn singularity at high temperature, due to the exponential growth with mass level of the number of modes that can be thermalized. This makes the thermal string scenario be applicable only at the moment when temperature becomes low enough. We call the previous cosmological era the Hagedorn era. Actually since the Hagedorn singularity is the signal that the string theory is at the point of undergoing a phase transition, the cosmology does not break down in Hagedorn era but should be described by another phase of string theory. It is expected that a dynamical description of this Hagedorn phase transition in cosmology can lead to a solution to the initial Big-Bang singularity. In fact it is realized in [11] that toroidal type II compactifications in presence of "gravito-magnetic" fluxes lead to thermal models, free of Hagedorn-like divergences. The induced cosmological evolutions include bouncing [12, 13] or emerging universes [14], where no initial singularity is encountered, and the model remains in a perturbative regime.

At the exit of Hagedorn era the cosmological evolutions carry some common features conforming to sensible phenomenology. *i*) The effective potential lifts flat directions, and moduli can be stabilized at the local minima, obtaining time dependent masses. *ii*) The spontaneous supersymmetry breaking scale M drops proportionally with temperature, generating the hierarchy $M \ll M_{\text{Plank}}$. *iii*) Dilaton can always be controlled at small values so that the perturbative computation is always valid. *iv*) The evolution is that of a radiation dominated universe, although the coherent motion of M store an amount of energy proportional to that of radiation. Therefore this pattern of evolution is named as "radiation-like". *v*) Such radiation-like evolution is insensitive to the initial conditions at the exit of the Hagedorn era, and is the result of dynamical attraction.

The radiation-like evolution stops at some moment, and it has to since otherwise the supersymmetry scale would drop to zero, leading to the restoration of supersymmetry. In fact at the electroweak scale Λ_{ew}, infrared effects become relevant, which destabilizes the Higgs potential, triggering the electroweak phase transition, and meanwhile stabilizes the supersymmetry breaking scale at about Λ_{ew}. Soon after, matter formation takes place and the universe enters into the era of standard cosmology. It should be mentioned that when the supersymmetry breaking scale is stabilized at about Λ_{ew}, a cosmological constant of order M^4 will be generated (c.f. Eq.(6.50)). Thus the thermal string scenario has not yet able to explain the small cosmological constant.

Moduli stabilization

Cosmological solutions that are phenomenologically viable should have moduli stabilized. The last two chapters are concentrated on this aspect. The goal is to figure out local minima of the effective potential induced by thermal/quantum effects, since it is shown in Chapter 6 that such minima can stabilize moduli in the intermediate era. This issue has been studied in previous works where only perturbative effects were investigated. The works presented in this thesis attempt to identify non-perturbative effects.

We first considered toroidally compactified heterotic and type I superstrings at finite temperature. In the heterotic picture, it is shown that all internal moduli except the dilaton are dynamically stabilized at points with enhanced gauge symmetry. The latter are due to perturbative F-string states. The subtlety for dilaton is that in $D \geq 5$, the dilaton asymptotes to a constant value, while in $D = 4$, the dilaton turns out to have a logarithmically decreasing behavior.

Applying the type I/heterotic S-duality maps, we inferred novel contributions to the free energy of a gas of type I superstrings, which are due to BPS D-string wrapping internal compact directions. These D-string states become massless at certain points of moduli space, enhance the gauge group and lift flat directions in the closed string sector. This is in contrast to the result from naive perturbative computation in the type I string, where perturbative effects can only lead to moduli stabilization in the open string sector, while in the closed string sector, moduli are always flat directions, and no perturbative states can induce enhanced gauge symmetry. Thus cosmological evolution dynamically attract the system to enhanced symmetry points, where the enhanced gauge symmetries induced by non-perturbative effects should be treated on equal footing with those induced by perturbative effects.

In particular, the type I moduli stabilization inferred from heterotic side should be discussed for different dimensions. For $D \geq 7$, all type I internal moduli can be stabilized at strong coupling, and the dilaton is fixed at some value in the flat direction. For $D = 6$, the S-duality maps the heterotic string coupling into the type I volume modulus. It turns out that the type I dilaton is stabilized at the value corresponding to weak coupling regime, while the internal volume modulus asymptotes to some value in the flat direction. In case $D = 5$, all type I internal moduli can be stabilized at weak coupling, while the dilaton is frozen in the flat direction. Finally for $D = 4$, the internal moduli can be stabilized while type I dilaton inherits the logarithmic behavior from the heterotic side, and the dilaton motion drives the system deeper into weak coupling regime. It should be stressed that the effects of the massless BPS non-perturbative D-strings persist at weak coupling, as their masses are protected by supersymmetry. However since these D-string states are in strong coupling regime, their exact contributions to the free energy density is not clear, and it is expected that a direct computation of E1-instanton in the weak coupling regime of type I string can answer to this problem.

In the second work, we address the question of moduli stabilization in the context of type II superstring theory compactified on CY_3's. Flat directions of the classical potential exist, which can be organized as a product of special Kähler and quaternionic manifolds, as follows from $\mathcal{N}_4 = 2$ local supersymmetry. These moduli spaces admit singular loci, where the internal manifold has 2-cycles or 3-cycles collapse, rendering generically massive supermultiplets massless. We examined, in type IIA description, the cases where BPS D2-branes wrapping vanishing 2-cycles lead to hypermultiplets charged under $U(1)$ factors at conifold loci [20], or $SU(N)$ enhanced gauge symmetries coupled to g hypermultiplets at some "non-Abelian loci" [46]. We show that in the weak coupling regime, moduli are attracted at such particular points in the thermal string cosmology.

At tree level we "integrate in" the above non-perturbative light states in the effective supergravity action. This repairs IR divergence in the Wilsonian effective action. The one-loop Colemann-Weinberg effective potential is computed based on the tree level action using field theory method, and the result depends on the masses of the light non-perturbative states. This computation is valid when the temperature is low enough. Local potential minima arise precisely where the light fields become massless.

The scalars that are stabilized are those involved in the gauge theories geometrically engineered in the vicinities of the loci where the internal CY_3 is singular. In type IIA description on general grounds, all Kähler moduli can be stabilized at the locus corresponding to the tip of the Kähler cone. It is because all 2-cycles in H^{11} shrink to zero size at this point and therefore the states from D2-brane wrapping the shrinking spheres become massless. On the other hand the mechanism that we study can stabilize the complex structure moduli associated to shrinking 3-cycles that can be desingularized by blowing up. The universal hypermultiplet cannot be stabilized by this mechanism since it is not associated the geometrical deformation of the CY_3. We have shown an example of $h_{11} = 2$ and $h_{12} = 128$, in which the whole Kähler moduli space is lifted near the intersection of a conifold locus and an $SU(2)$ enhanced symmetry locus. Also some of the complex structure moduli are stabilized.

Perspective

We are still left with plenty of unsolved problems in the thermal string cosmology. Based on the work on type II strings in Chapter 8, we would like to extend the analysis to the type II models compactified on generalized Calabi-Yau spaces [69], including fluxes, branes or orientifold projections. We expect that the presence of these extra objects and structures can provide novel mechanisms for moduli stabilization. Moreover this setup leads to $\mathcal{N}_4 = 1$ backgrounds which is phenomenologically more viable.

We can further consider its extrapolation into the Hagedorn era, implementing the "gravito-magnetic" fluxes [11] in (generalized) Calabi-Yau compactifications of type II strings and it will be interesting to see if this can give a theoretical framework able to account for very early cosmological time. In addition to addressing the problems of dynamical description of the Hagedorn phase transition and the resolution of the Big-Bang singularity, another significant aspect in the Hagedorn era is the possibility of achieving an alternative to the inflationary scenario in the early universe.

Moreover it is always of great interest to investigate the standard cosmology era, in order to make connection to the observable universe. For this we need a background with $\mathcal{N}_4 = 1$ supersymmetry spontaneously broken at zero temperature, which is necessary to obtain MSSM-like model containing chiral matter. For exact computation, the fermionic contribution of heterotic string models [35] can be an efficient tool for constructing such models. We can carry out, in such context, the analysis in [9, 10] to unravel the infrared effects that can stabilize the supersymmetry breaking scale, and the work is in progress. After stabilizing the supersymmetry breaking scale, moduli acquire real physical masses which are constant and meanwhile the electroweak phase transition takes place and the standard matter content starts to form. It is only after this moment that we can address the problem concerning the standard observable universe such as the problem of dark matter.

Appendix A

Computation of canonical partition function of ideal gas

The goal of this appendix is to show all the technical details that Chapter 5 is based on. In the first two sections we will compute the partition function $\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H}$ for an ideal gas of point particles, showing the equivalence of the results expressed in different forms. In the third section we include mathematical formulae useful in the computation.

A.1 Second quantization computation

The second quantized action describing one particle degree of freedom in the Euclidean space of dimension D is

$$S = \int d^D x \, \phi \left(-\Box_{\mathrm{E}} + M^2 \right) \phi, \tag{A.1}$$

where $\Box_{\mathrm{E}} = \partial_0^2 + \vec{\nabla}^2$, and M is the mass. Note that this action can also describe one fermion degree of freedom where ϕ is a real grassmanian variable. The canonical partition function $\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H}$ can be evaluated by the path integral

$$\ln \mathcal{Z}_{\mathrm{B,F}} = \ln \left(\int \mathcal{D}\phi \; e^{-S[\phi]} \right) = \ln \left[\mathrm{Det} \left(-\Box_{\mathrm{E}} + M^2 \right) \right]^{\mp 1/2} = \mp \frac{1}{2} \mathrm{Tr} \ln \left(-\Box_{\mathrm{E}} + M^2 \right). \tag{A.2}$$

with the Euclidean time compactified on a circle of radius $R_0 = \beta/2\pi$, and wave function taking periodic boundary condition for boson and anti-periodic boundary condition for fermion. Here we derive the results in the two distinct forms that are used in Chapter 5.

Standard formulae of canonical ensemble

Thus for a bosonic degree of freedom

$$\begin{aligned}\ln \mathcal{Z}_{\rm B} &= -\frac{1}{2}{\rm Tr}\ \ln\left(-\Box_{\rm E} + M^2\right) = -\frac{V_{D-1}}{2(2\pi)^{D-1}}\sum_m \int d\vec{p}\ \ln\left(\frac{m^2}{R_0^2} + \vec{p}^{\,2} + M^2\right)\\ &= -\frac{V_{D-1}}{2(2\pi)^{D-1}}\sum_m \int d\vec{p}\ \ln\left(\frac{m^2}{R_0^2} + \omega_p^2\right)\\ &= -\frac{V_{D-1}}{2(2\pi)^{D-1}}\int d\vec{p}\left[\ln\prod_{m\in\mathbb{Z}}\omega_p^2 + \ln\prod_{m\in\mathbb{Z}}\left(\frac{m^2}{R_0^2} + \omega_p^2\right)\right],\end{aligned} \tag{A.3}$$

where $\omega_p = \sqrt{\vec{p}^{\,2} + M^2}$. Using Eqs.(A.44) and (A.46) it is not hard to see that $\prod_m \omega_p^2 = 1$ and $\prod_m \left(\frac{m^2}{R_0^2\omega_p^2} + 1\right) = 4\sinh^2(\pi R_0\,\omega_p)$. Therefore resuming the calculation in Eq.(A.3),

$$\begin{aligned}\ln \mathcal{Z}_{\rm B} &= -\frac{V_{D-1}}{2(2\pi)^{D-1}}\int d\vec{p}\ \ln\left[4\sinh^2\left(\pi R_0\,\omega_p\right)\right]\\ &= -\frac{V_{D-1}}{(2\pi)^{D-1}}\int d\vec{p}\ \left[\frac{1}{2}\beta\omega_p + \ln\left(1 - e^{-\beta\omega_p}\right)\right].\end{aligned} \tag{A.4}$$

For a fermionic degree of freedom, we have

$$\begin{aligned}\ln \mathcal{Z}_{\rm F} &= \frac{1}{2}{\rm Tr}\ \ln\left(-\Box_{\rm E} + M^2\right) = \frac{V_{D-1}}{2(2\pi)^{D-1}}\sum_m \int d\vec{p}\ \ln\left[\frac{(m+1/2)^2}{R_0^2} + \vec{p}^{\,2} + M^2\right]\\ &= \frac{V_{D-1}}{2(2\pi)^{D-1}}\int d\vec{p}\left\{\ln\prod_{m\in\mathbb{Z}}\omega_p^2 + \ln\prod_{m\in\mathbb{Z}}\left[\frac{(m+1/2)^2}{R_0^2} + \omega_p^2\right]\right\}.\end{aligned} \tag{A.5}$$

Using the formula (A.49), we have

$$\begin{aligned}\ln \mathcal{Z}_{\rm F} &= \frac{V_{D-1}}{2(2\pi)^{D-1}}\int d\vec{p}\ \ln\left[4\cosh^2\left(\pi R_0\,\omega_p\right)\right]\\ &= \frac{V_{D-1}}{(2\pi)^{D-1}}\int d\vec{p}\left[\frac{1}{2}\beta\omega_p + \ln\left(1 + e^{-\beta\omega_p}\right)\right].\end{aligned} \tag{A.6}$$

Schwinger parameter representation

We can obtain an equivalent form of the above results, by using the integral representation of the logarithmic function Eq.(A.52) in evaluating the path integral Eq.(A.2).

$$\begin{aligned}\ln \mathcal{Z}_{\rm B} &= -\frac{1}{2}{\rm Tr}\ \ln\left(-\Box_{\rm E} + M^2\right) = \frac{1}{2}{\rm Tr}\ \int_0^\infty \frac{d\ell}{\ell}\left[e^{-\ell\left(-\Box_{\rm E} + M^2\right)} - e^{-\ell}\right]\\ &= \frac{V_{D-1}}{2(2\pi)^{D-1}}\sum_m \int d\vec{p}\int_0^\infty \frac{d\ell}{\ell}\left[e^{-\ell\left(\frac{m^2}{R_0^2} + \vec{p}^{\,2} + M^2\right)} - e^{-\ell}\right],\end{aligned} \tag{A.7}$$

where ℓ is the Schwinger parameter, whose physical interpretation is the proper time that a particle spends to close up a loop trajectory. This will be better clarified in the first quantization computation in Sec.A.2. The second term in the bracket above, which integrates up to infinity, is completely irrelevant since it contains no physical information. Therefore it can be discarded naively. In fact the Riemann-Zeta regularization of the sum over m also yields zero. Then we integrate over the spatial momentum $\vec{p}$ using Eq.(A.51) and Poisson resum over m using Eq.(A.50), and we get

$$\begin{aligned}\ln \mathcal{Z}_{\rm B} &= \frac{\beta V_{D-1}}{2(4\pi)^{D/2}} \sum_m \int_0^\infty \frac{d\ell}{\ell^{1+D/2}} \exp\Big(-\frac{\pi^2 R_0^2}{\ell} m^2 - \ell M^2\Big) \\ &= \frac{\beta V_{D-1}}{2(2\pi)^D} \sum_m \int_0^\infty \frac{d\ell}{\ell^{1+D/2}} \exp\Big(-\frac{\pi R_0^2}{\ell} m^2 - \pi \ell M^2\Big),\end{aligned} \tag{A.8}$$

where in the second line we have simply recaled the Schwinger parameter, so that the expression is more adapted to string theory language. The case of one fermion degree of freedom proceeds similarly, which yields the fermionic counterpart of Eq.(A.7)

$$\begin{aligned}\ln \mathcal{Z}_{\rm F} &= \frac{1}{2}{\rm Tr}\, \ln\big(-\Box_{\rm E} + M^2\big) = -\frac{1}{2}{\rm Tr} \int_0^\infty \frac{d\ell}{\ell}\Big[e^{-\ell(-\Box_{\rm E}+M^2)} - e^{-\ell}\Big] \\ &= -\frac{V_{D-1}}{2(2\pi)^{D-1}} \sum_m \int d\vec{p} \int_0^\infty \frac{d\ell}{\ell}\left\{e^{-\ell\left[\frac{(m+1/2)^2}{R_0^2}+\vec{p}^2+M^2\right]} - e^{-\ell}\right\},\end{aligned} \tag{A.9}$$

where the shift on the summing index $m+1/2$ is due to the anti-periodic boundary condition. Then we remove the physically irrelevant infinity, and now the Poisson resummation and the integral over $\vec{p}$ gives

$$\begin{aligned}\ln \mathcal{Z}_{\rm F} &= \frac{\beta V_{D-1}}{2(4\pi)^{D/2}} \sum_m \int_0^\infty \frac{d\ell}{\ell^{1+D/2}} (-1)^{m+1} \exp\Big(-\frac{\pi^2 R_0^2}{\ell} m^2 - \ell M^2\Big) \\ &= -\frac{\beta V_{D-1}}{2(2\pi)^D} \sum_m \int_0^\infty \frac{d\ell}{\ell^{1+D/2}} (-1)^m \exp\Big(-\frac{\pi R_0^2}{\ell} m^2 - \pi \ell M^2\Big).\end{aligned} \tag{A.10}$$

Equivalence check

The above two formalisms, Eqs (A.4,A.6) and Eqs (A.8,A.9) are obtained through different ways and have nothing to do with each other by appearance. Here we show directly that they are the same thing up to a physically irrelevant infinity. Using Eq.(A.53), we have, for any $n \neq 0$,

$$\frac{1}{(2\pi)^D} \int_0^\infty \frac{d\ell}{\ell^{1+D/2}} \exp\Big(-\frac{\pi R_0^2}{\ell} n^2 - \pi\, \ell\, M^2\Big) = 2\Big(\frac{M}{2\pi n \beta}\Big)^{\frac{D}{2}} K_{\frac{D}{2}}\big(n\beta M\big), \tag{A.11}$$

On the other hand, using Eqs.(A.54) and (A.58), we have

$$\frac{1}{(2\pi)^{D-1}}\int d\vec{p}\; e^{-n\beta\omega_p} = 2\beta n\Big(\frac{M}{2\pi n\beta}\Big)^{\frac{D}{2}} K_{\frac{D}{2}}\big(n\beta M\big). \tag{A.12}$$

Thus combining the above two equations, we get

$$\frac{\beta}{(2\pi)^D}\int_0^\infty \frac{d\ell}{\ell^{1+D/2}}\exp\Big(-\frac{\pi R_0^2}{\ell}n^2 - \pi\,\ell\, M^2\Big) = \frac{1}{(2\pi)^{D-1}}\int d^{D-1}p\;\frac{e^{-n\beta\omega_p}}{n}, \tag{A.13}$$

for $n > 0$. Summing up the two sides over n, we have

$$\begin{aligned}
&\frac{\beta}{2(2\pi)^D}\sum_{m\neq 0}\int_0^\infty \frac{d\ell}{\ell^{1+D/2}}\exp\Big(-\frac{\pi R_0^2}{\ell}m^2 - \pi\,\ell\, M^2\Big)\\
&\qquad = \frac{1}{(2\pi)^{D-1}}\sum_{n>0}\int d\vec{p}\;\frac{e^{-n\beta\omega_p}}{n} = -\frac{1}{(2\pi)^{D-1}}\int d\vec{p}\;\ln\big(1-e^{-\beta\omega_p}\big);
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
&\frac{\beta}{2(2\pi)^D}\sum_{m\neq 0}\int_0^\infty \frac{d\ell}{\ell^{1+D/2}}(-1)^{m+1}\exp\Big(-\frac{\pi R_0^2}{\ell}m^2 - \pi\,\ell\, M^2\Big)\\
&\qquad = \frac{1}{(2\pi)^{D-1}}\sum_{n>0}\int d\vec{p}\;(-1)^{n+1}\frac{e^{-n\beta\omega_p}}{n} = \frac{1}{(2\pi)^{D-1}}\int d\vec{p}\;\ln\big(1+e^{-\beta\omega_p}\big).
\end{aligned} \tag{A.15}$$

With the above equalities we are almost allowed to say that Eqs(A.4) and (A.6) are equivalent to Eqs (A.7) and (A.9), but we still have to sort out the vacuum bubble contribution. Then we examine the zero modes in Schwinger parameter representation, that is the $m = 0$ term in Eq.(A.8) or (A.10). We regard the zero mode as the limit $n \to 0$ of Eq.(A.13), and therefore

$$\begin{aligned}
\pm\frac{\beta V_{D-1}}{2(4\pi)^{D/2}}\int_0^\infty \frac{d\ell}{\ell^{1+D/2}}\exp\big(-\ell M^2\big) &= \pm\lim_{\epsilon\to 0}\frac{V_{D-1}}{2(2\pi)^{D-1}}\int d\vec{p}\;\frac{e^{-\epsilon\beta\omega_p}}{\epsilon}\\
&= \pm\lim_{\epsilon\to 0}\frac{V_{D-1}}{(2\pi)^{D-1}}\int d\vec{p}\Big[-\frac{1}{2}\beta\omega_p + \frac{1}{\epsilon} + \mathcal{O}(\epsilon)\Big]\\
&= \mp\frac{V_{D-1}}{(2\pi)^{D-1}}\int d\vec{p}\,\frac{1}{2}\beta\omega_p + (\text{irrelevant infinity})
\end{aligned} \tag{A.16}$$

where the last line is just the vacuum bubble contribution in the standard formalism with a non physical infinity. This completes the equivalence check

A.2 First quantization computation

In Sec.5.1 we stated that $\ln\mathcal{Z} = \bigcirc$, which can be evaluated by first quantization formalism. Here we perform the explicit calculation. The action describing a relativistic free particle of mass M in the Euclidean space is[1]

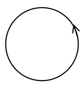
$$S = \frac{1}{2}\int_0^1 dt\sqrt{g}\left(g^{tt}\,\dot{x}^2 + M^2\right), \tag{A.17}$$

where t is the Euclidean proper time parameterizing the worldline of the particle, and $g_{tt} = g_{tt}(t)$ the metric of the worldline. The target Euclidean space is taken to be Cartesian of dimension D, and in the above action, $\dot{x}^2$ actually means $\delta_{ij}\dot{x}^i\dot{x}^j$. For simplicity in the following contents, we keep implicit the sum or the product over the Euclidean space dimension. Often one takes $g_{tt} = e^2$ with $e > 0$, so that we have the very familiar form

$$S = \frac{1}{2}\int_0^1 dt\Big(e^{-1}\,\dot{x}^2 + e\,m^2\Big). \tag{A.18}$$

This action is invariant under diffeomorphisms, which is the gauge symmetry of the system: $\delta x = \epsilon$, $\delta g_{tt} = 2g_{tt}\nabla_t\,\epsilon$, where ∇_t is the covariant derivative with Levi-Civita coefficient $\Gamma^t_{\ tt} = \frac{1}{2}\partial_t \ln g_{tt}$.

The loop amplitude, in terms of path integral, is



$$= \int \frac{\mathcal{D}g\,\mathcal{D}x}{\mathrm{Vol(Diff)}}\, e^{-S[g,x]}, \tag{A.19}$$

where $\mathcal{D}x = \mathcal{D}x^0 \ldots \mathcal{D}x^{D-1}$ is understood. Since the topology of the particle trajectory is a circle instead of a segment, the diffeomorphism group contains an isometry subgroup generated by Killing vectors. This isometry gauge redundancy is clearly missed by the path integral over the metric $\int \mathcal{D}g$, since when performing this path integral, each different value of $g_{\tau\tau}$ is counted only once. However one still needs to divide by the volume of the whole diffeomorphism group Vol(Diff) to remove the gauge redundancy, because the isometry redundancy missed by $\int \mathcal{D}g$ is actually picked up by $\int \mathcal{D}x$. Another subtlety is that all the possible metrics g_{tt} are not on the same gauge orbit. Distinct gauge orbits are labeled by the Teichmüler parameters which for our case, is the perimeter of the circle: $\ell = \int_0^1 dt\sqrt{g} > 0$. By definition, ℓ is gauge invariant, so that two gauge orbits of different ℓ never intersect each other, and also, any arbitrary metric g_{tt} can be transformed into a constant one equal to its Teichmüler parameter squared: $\hat{g}_{tt} = \ell^2$, by a diffeomorphism.

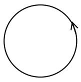
Now we perform gauge fixing using Polchinski's technique. We insert in the path integral (A.19) a constant functional of g_{tt}:

$$1 = \Delta_{\mathrm{FP}}(g)\int_0^\infty d\ell \int \mathcal{D}'\xi\,\delta\big(g_{tt} - \mathcal{T}_\xi\,\ell^2\big), \tag{A.20}$$

where $\mathcal{T}_\xi$ denotes the diffeomorphism generated by the vector field $\xi(t)$, ℓ^2 is regarded as a constant metric, and the path integral $\int \mathcal{D}'\xi$ excludes the zero mode of ∇_t, that is, the Killing vectors. Also

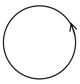
[1]Throughout this section S always denote the action in first quantization formalism, not to be confused with the action Eq.(A.1) in second quantization formalism.

it can be shown that the Faddeev-Popov determinant is gauge invariant: $\Delta_{\rm FP}(\mathcal{T}_\eta\, g) = \Delta_{\rm FP}(g)$. This insertion gives



$$
\begin{aligned}
&= \int \frac{\mathcal{D}g\,\mathcal{D}x}{\mathrm{Vol(Diff)}}\,\Delta_{\rm FP}(g)\int_0^\infty d\ell \int \mathcal{D}'\xi\,\delta\big(g_{tt} - \mathcal{T}_\xi \ell^2\big)\,e^{-S[g,x]}.\\
&= \int_0^\infty d\ell \int \frac{\mathcal{D}x\,\mathcal{D}'\xi}{2\mathrm{Vol(Diff_0)}}\,\Delta_{\rm FP}(\mathcal{T}_\xi\ell^2)\,e^{-S[\mathcal{T}_\xi\ell^2,\,x]}
\end{aligned}
\tag{A.21}
$$

Passing to the second line, we have only performed the integral $\int \mathcal{D}g$, which is calculable thanks to the presence of the δ-functional. Also in the second line, Diff_0 denotes the connected component of the diffeomorphism group containing the identity element. The rest of the diffeomorphism group $\mathrm{Diff} - \mathrm{Diff}_0$ is also a connected component, which is simply the composition of Diff_0 to a time reversal in the worldline $t \to 1 - t$. Thus we have $\mathrm{Vol(Diff)} = 2\,\mathrm{Vol(Diff_0)}$. A very important point here is that $\mathrm{Vol(Diff_0)}$ can depend on the Teichemüler parameter, which we will see very soon, so that it is put inside the integration over ℓ. From the volume $\mathrm{Vol(Diff_0)}$ we can further factorize out the volume generated by isometries: $\mathrm{Vol(Diff_0)} = \mathrm{Vol(Diff_0^\perp)}\mathrm{Vol(KV_0)}$, where $\mathrm{Diff}_0^\perp$ is the subgroup generated by non-Killing vectors whose volume satisfies $\int \frac{\mathcal{D}'\xi}{\mathrm{Vol(Diff_0^\perp)}} = 1$, and KV_0 is the subgroup of isometries. Therefore in Eq.(A.21), replacing the mute variable x with $\mathcal{T}_\xi x$, and making use of the gauge invariance of the action, the Faddeev-Popov determinent and the measure $\mathcal{D}x$, we obtain



$$
\begin{aligned}
&= \frac{1}{2}\int_0^\infty d\ell \int \frac{\mathcal{D}\big(\mathcal{T}_\xi x\big)\,\mathcal{D}'\xi}{\mathrm{Vol(Diff_0^\perp)Vol(KV_0)}}\,\Delta_{\rm FP}(\mathcal{T}_\xi\ell^2)\,e^{-S[\mathcal{T}_\xi\ell^2,\,\mathcal{T}_\xi x]}\\
&= \frac{1}{2}\int_0^\infty d\ell \int \frac{\mathcal{D}\big(\mathcal{T}_\xi x\big)\,\mathcal{D}'\xi}{\mathrm{Vol(Diff_0^\perp)Vol(KV_0)}}\,\Delta_{\rm FP}(\mathcal{T}_\xi\ell^2)\,e^{-S[\ell^2,\,x]}\\
&= \frac{1}{2}\int_0^\infty d\ell \int \frac{\mathcal{D}x}{\mathrm{Vol(KV_0)}}\,\Delta_{\rm FP}(\ell^2)\,e^{-S[\ell^2,\,x]}
\end{aligned}
\tag{A.22}
$$

Now we proceed to evaluate the Faddeev-Popov determinant $\Delta_{\rm FP}$. Generically the metric variation based on a reference metric, following a diffeomorphism ξ and a variation $\delta\ell$ in the Teichmüler parameter reads

$$
\delta g_{tt} = 2\,g_{tt}\,\nabla_t\xi + \big(\partial_\ell\, g_{tt}\big)\,\delta\ell = 2\,\nabla_t\big(g_{tt}\,\xi\big) + \big(\partial_\ell\, g_{tt}\big)\,\delta\ell\,.
\tag{A.23}
$$

Thus the Faddeev-Popov determinant has the integral representation

$$
\begin{aligned}
\Delta_{\rm FP}(g_{tt}) &= \int \mathcal{D}b\,\mathcal{D}'c\,d\lambda\,\exp\Big\{\int_0^1 dt\sqrt{g}\,b\Big[2\,\nabla_t c + \big(\partial_\ell\, g_{tt}\big)\lambda\Big]\Big\}\\
&= \int \mathcal{D}b\,\mathcal{D}'c\,\big(b, \partial_\ell\, g_{tt}\big)\exp\Big(2\int_0^1 dt\sqrt{g}\,b\,\nabla_t c\Big).
\end{aligned}
\tag{A.24}
$$

In the above expression, λ is a grassmannian variable, while b and c are the anti-ghost and the ghost fields with tensor structures $(2,0)$ and $(0,1)$ respectively. The path integral over c does not include the zero mode of ∇_t. From the first line to the second line, we have integrated over λ, with $\big(b, \partial_\ell g_{tt}\big)$ the shorthand notation for $\int_0^1 dt \sqrt{g}\, b\, \partial_\ell g_{tt}$. To compute Eq.(A.24) explicitly, we take the gauge $g_{tt} = \ell^2$, so that $\partial_\ell g_{tt} = 2\ell$, $\nabla_t c = \partial_t c$, and therefore Eq.(A.24) becomes

$$\Delta_{\rm FP}(\ell^2) = \int \mathcal{D}b\, \mathcal{D}'c \,\big(b, 2\,\ell\big) \exp\Big(2\int_0^1 dt\, \ell\, b\, \partial_t c\Big). \tag{A.25}$$

Now we choose the orthonormal basis functions with periodical boundary condition

$$\phi_k(t) = \ell^{-\frac{1}{2}} e^{2\pi i\, k\, t} \quad \text{where} \quad k = 0, \pm 1, \pm 2, \ldots, \tag{A.26}$$

and expand the ghost fields against this basis:

$$c(t) = \sum_{k\neq 0} c_k\, \phi_k(t), \quad b(t) = \sum_k b_k\, \phi_k(t), \quad \text{where } \; c_{-k} = c_k^\dagger \; \text{ and } \; b_{-k} = b_k^\dagger\,. \tag{A.27}$$

To obtain the integration measures of them, we compute the norms of b and c as if they were non-grassmannian:

$$||b||^2 = \int_0^1 dt \sqrt{g}\, g_{tt} g_{tt}\, b^2 = \ell^4 \sum_k \big|b_k\big|^2, \tag{A.28}$$

$$||c||^2 = \int_0^1 dt \sqrt{g}\, g^{tt}\, c^2 = \ell^{-2} \sum_{k\neq 0} \big|c_k\big|^2. \tag{A.29}$$

This yields the integration measures

$$\mathcal{D}b = \prod_k \frac{db_k}{\ell^2} = \prod_k db_k, \tag{A.30}$$

$$\mathcal{D}'c = \prod_{k\neq 0} \ell^2 dc_k = \ell^{-1/2} \prod_{k\neq 0} dc_k\,. \tag{A.31}$$

Here we have regularized with $\prod_{n=1}^\infty a = a^{-1/2}$. We also compute easily

$$\big(b, 2\,\ell\big) = \int_0^1 dt\, 2\, \ell^{3/2} b = 2\, \ell^{3/2}\, b_0, \tag{A.32}$$

$$2\int_0^1 dt\, \ell\, b\, \partial_t c = \sum_k 4\pi i\, k\, b_{-k} c_k. \tag{A.33}$$

Thus resuming Eq.(A.25), we get

$$\begin{aligned}
\Delta_{\rm FP}(\ell^2) &= \int \mathcal{D}b\, \mathcal{D}'c \,\big(b, 2\,\ell\big) \exp\Big(2\int_0^1 dt\, \ell\, b\, \partial_t c\Big) \\
&= 2\sqrt{\ell} \int db_0\, b_0 \prod_{k\neq 0} \int dc_k db_{-k} \exp\big(4\pi i\, k\, b_{-k} c_k\big) \\
&= 2\sqrt{\ell} \prod_{k\neq 0} 4\pi i\, k = 2\sqrt{\ell} \prod_{n=1}^{\infty} \big(16\pi^2 n^2\big) = \sqrt{\ell}.
\end{aligned} \tag{A.34}$$

Here we used the regularization $\prod_{n=1}^{\infty} n^s = (2\pi)^{s/2}$. We can perform the same trick to compute the path integral of $\mathcal{D}x$ in Eq.(A.22). Separating the quantum fluctuation from the classical solutions $x(t) = x_{\rm cl}(t) + \delta x(t)$, expanding the quantum fluctuation against the orthonormal basis (A.26): $\delta x(t) = \sum_k \delta x_k\, \phi_k(t)$ with $\delta x_k = \delta x^*_{-k}$, we compute the norm in terms of the coefficients

$$\|\delta x\|^2 = \delta x_0^2 + 2\sum_{n=1}^{\infty}\Big[\big(\delta x_n^{\rm Re}\big)^2 + \big(\delta x_n^{\rm Im}\big)^2\Big]. \tag{A.35}$$

The target Euclidean space dimension is summed over implicitly. The integration measure for δx is

$$\mathcal{D}\delta x = dx_0 \prod_{n=1}^{\infty} 2dx_n^{\rm Re} dx_n^{\rm Im} = \frac{1}{2^{D/2}} dx_0 \prod_{n=1}^{\infty} dx_n^{\rm Re} dx_n^{\rm Im}, \tag{A.36}$$

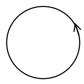
where again, the indices denoting the target space dimensions are not explicitly written out. On the other hand, the point-particle action after gauge fixing becomes

$$S[\ell^2, x] = S[\ell^2, x_{\rm cl}] + \frac{1}{2}\int_0^1 dt\, \ell^{-1}\big(\delta\dot{x}\big)^2 = S[\ell^2, x_{\rm cl}] + \sum_{n=1}^{\infty}\Big(\frac{2\pi n}{\ell}\Big)^2\Big[\big(\delta x_n^{\rm Re}\big)^2 + \big(\delta x_n^{\rm Im}\big)^2\Big]. \tag{A.37}$$

Thus we are able to evaluate the path integral

$$\begin{aligned}\int \mathcal{D}x\, e^{-S[\ell^2, x]} &= \sum_{x_{\rm cl}} e^{-S[\ell^2, x_{\rm cl}]} \int \mathcal{D}\delta x \exp\Big[-\frac{1}{2}\int dt\, \ell^{-1}\big(\delta\dot{x}\big)^2\Big] \\ &= \sum_{x_{\rm cl}} e^{-S[\ell^2, x_{\rm cl}]} \frac{1}{2^{D/2}} \int dx_0 \prod_{n=1}^{\infty} \int dx_n^{\rm Re} dx_n^{\rm Im} \exp\Big\{-\Big(\frac{2\pi n}{\ell}\Big)^2\Big[\big(\delta x_n^{\rm Re}\big)^2 + \big(\delta x_n^{\rm Im}\big)^2\Big]\Big\} \\ &= \sum_{x_{\rm cl}} e^{-S[\ell^2, x_{\rm cl}]} V_D\Big(\frac{\ell}{2}\Big)^{D/2} \prod_{n=1}^{\infty}\Big(\frac{\ell^2}{4\pi\, n^2}\Big)^D = \frac{V_D}{(2\pi\ell)^{D/2}} \sum_{x_{\rm cl}} e^{-S[\ell^2, x_{\rm cl}]}. \end{aligned} \tag{A.38}$$

Here since the zero mode coefficent δx_0 is with respect to the basis $1/\sqrt{\ell}$, its integration gives the target space volume V_D dressed by $\ell^{D/2}$. The last element in Eq.(A.22) to evaluate is the isometry group volume $\mathrm{Vol(KV_0)}$. In the gauge $g_{tt} = \ell^2$, the killing vectors are just the vector with constant component $\xi_0 = \text{const.} \in [0, 1)$, and the isometry group volume is the path integral $\mathrm{Vol(KV_0)} = \int \mathcal{D}\xi_0$. The norm of such Killing vectors is $\|\xi_0\|^2 = \int_0^1 dt \sqrt{g}\, g_{tt}\xi_0^2 = \ell^3\xi_0^2$, and therefore the path integral is evaluated as $\mathrm{Vol(KV_0)} = \int_0^1 \ell^{3/2} d\xi_0 = \ell^{3/2}$. Plugging this result, the Faddeev-Popov determinant Eq.(A.34), and the path integral Eq.(A.38) into Eq.(A.22), we obtain the result for the loop amplitude



$$= \frac{V_D}{2(2\pi)^{D/2}} \int_0^{\infty} \frac{d\ell}{\ell^{D/2+1}} \sum_{x_{\rm cl}} \exp\Big(-S[\ell^2, x_{\rm cl}]\Big). \tag{A.39}$$

Finally we switch on finite temperature T, where the Euclidian time is compactified on a circle of radius R_0. The inverse temperature is just the perimeter of the circle $\beta = 2\pi R_0$. The classical solutions are those describing the particle circulating the time circle:

$$x^0_{\rm cl}(t) = 2\pi\, m\, R_0\, t, \quad x^{i\ge 1}_{\rm cl} \equiv \text{const.}, \quad \text{where} \;\; m = 0, \pm 1, \pm 2, \dots \tag{A.40}$$

Thus $S[\ell^2, x_{\rm cl}] = \frac{2\pi^2 m^2 R_0^2}{\ell} + \frac{\ell M^2}{2}$, and the sum over the classical solutions becomes the sum over m. Thus taking into account the different boundary conditions for boson and fermion, Eq.(A.39) becomes

$$\bigcirc = \frac{\beta V_{D-1}}{2(2\pi)^{D/2}} \int_0^\infty \frac{d\ell}{\ell^{D/2+1}} \sum_m (-1)^{Fm+F} \exp\Big(- \frac{2\pi^2 m^2 R_0^2}{\ell} - \frac{\ell M^2}{2} \Big). \tag{A.41}$$

with $F = 0$ for boson and $F = 1$ for fermion. To switch to string theory language, we rescale $\ell \to 2\pi\ell$, and obtain

$$\bigcirc = \frac{\beta V_{D-1}}{2(2\pi)^{D}} \int_0^\infty \frac{d\ell}{\ell^{D/2+1}} \sum_m (-1)^{Fm+F} \exp\Big(- \frac{\pi R_0^2}{\ell} m^2 - \pi\, \ell\, M^2 \Big), \tag{A.42}$$

which contains the correct overall coefficient $\frac{\beta V_{D-1}}{2(2\pi)^D}$ in string theories.

A.3 Mathematical formulae

⋄ Riemann-Zeta regularizations:

$$\prod_{n\ge 1} n^a = e^{-a\zeta'(0)} = (2\pi)^{a/2}; \tag{A.43}$$

$$\prod_{n\ge 1} a = a^{\zeta(0)} = a^{-1/2}. \tag{A.44}$$

⋄ Infinite product representation of (hyper-)trigonometric functions :

$$\prod_{k\in\mathbb{Z}} (k + a) = a \prod_{n\ge 1} \left(a^2 - n^2\right) = 2i \sin(\pi a). \tag{A.45}$$

$$\prod_{n\ge 1} \left(n^2 - a^2\right) = 2\, a^{-1} \sin(\pi a); \quad \prod_{n\ge 1} \left(n^2 + a^2\right) = 2\, a^{-1} \sinh(\pi a). \tag{A.46}$$

$$\prod_{k\in\mathbb{Z}} \Big(k \pm \frac{1}{2} + a\Big) = \pm 2i \cos(\pi a). \tag{A.47}$$

$$\prod_{k\in\mathbb{Z}} \Big[\Big(k + \frac{1}{2}\Big)^2 - a^2\Big] = \prod_{k\in\mathbb{Z}} \Big[\Big(k + \frac{1}{2} + a\Big)\Big(k - \frac{1}{2} + a\Big)\Big] = 4\cos^2(\pi a); \tag{A.48}$$

$$\prod_{k\in\mathbb{Z}} \Big[\Big(k + \frac{1}{2}\Big)^2 + a^2\Big] = 4\cosh^2(\pi a). \tag{A.49}$$

◇ Poisson resummation and Gaussian integral :

$$\sum_{\tilde{m}} e^{-\pi \tilde{m}^{\mathrm{T}} A \tilde{m} + 2\pi i b^{\mathrm{T}} \tilde{m}} = \frac{1}{\sqrt{\det A}} \sum_{m} e^{-\pi (m-b)^{\mathrm{T}} A^{-1} (m-b)}. \tag{A.50}$$

$$\int_{-\infty}^{\infty} dx \;\; \exp\left(-\frac{1}{2} x^{\mathrm{T}} A x\right) = \det\left(\frac{A}{2\pi}\right)^{-1/2}. \tag{A.51}$$

◇ Integral representation of logarithmic function :

$$\ln A = -\int_0^{\infty} \frac{d\ell}{\ell} \big(e^{-A\,\ell} - e^{-\ell}\big). \tag{A.52}$$

◇ Modified Bessel functions of the second kind :

$$\int_0^{\infty} \frac{d\ell}{\ell^{1+\nu}} \exp\Big(-\frac{A}{\ell} - B\,\ell\Big) = 2\left(\frac{B}{A}\right)^{\frac{\nu}{2}} K_{\nu}\big(2\sqrt{AB}\,\big), \tag{A.53}$$

$$\int_0^{\infty} p^n dp\, e^{-\beta\sqrt{p^2+M^2}} = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}}\, \Gamma\Big(\frac{n+1}{2}\Big)\, M \Big(\frac{M}{\beta}\Big)^{\frac{n}{2}} K_{\frac{n+2}{2}}(\beta M). \tag{A.54}$$

$$K_{\nu}(x) \sim 2^{\nu-1}\Gamma(\nu)\, x^{-\nu} - 2^{\nu-3}\Gamma(\nu-1)\, x^{2-\nu} + \ldots \quad (x \sim 0)\,; \tag{A.55}$$

$$K_{\nu}(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}} + \ldots \quad (x \sim \infty)\,. \tag{A.56}$$

◇ Others :

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\,. \tag{A.57}$$

$$d^n x = \frac{n\,\pi^{\frac{n}{2}}}{\Gamma\big(\frac{n}{2}+1\big)} x^{n-1} dx\,. \tag{A.58}$$

Appendix B

Equations of motion in cosmology

We derive the equations of motion of cosmology adapted to string theory language. Consider the Robertson-Walker metric containing a spatial curvature

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\Big[\frac{dr^2}{1+k\,r^2} + r^2 d\Omega^2_{D-2}\Big], \tag{B.1}$$

where $d\Omega^2_{D-2}$ is the metric of S^{D-2}, which can be expressed explicitly as

$$d\Omega^2_{D-2} = d\theta_1^2 + \sin^2\theta_2\big(d\theta_3 + \sin^2\theta_3\big(d\theta_4^2 + \dots\big)\big). \tag{B.2}$$

From the metric one deduces the Einstein tensor and the scalar curvature:

$$G_{00} = -\frac{1}{2}(D-1)(D-2)N^{-2}\big(H^2 + N^2 a^{-2} k\big)\,g_{00}, \tag{B.3}$$

$$G_{ij} = -\frac{1}{2}(D-2)N^{-2}\big[2\dot{H} - 2N^{-1}\dot{N} + (D-1)H^2 + (D-3)\,N^2 a^{-2} k\big]\,g_{ij}\,, \tag{B.4}$$

$$R = -\frac{2}{D-2}G = \big(D-1\big)N^{-2}\big[2\dot{H} + D\,H^2 + (D-2)N^2 a^{-2} k - 2N^{-1}\dot{N}H\big]. \tag{B.5}$$

We can then write down the action and derive the equations of motion. We split the total action into the gravitational part S_g and the matter part S_m, where

$$S_g = \int d^D x\sqrt{-g}\,s\,\frac{R}{2} = \frac{D-1}{2}\int d^D x\,\frac{s\,a^{D-1}}{N}\Big[-(D-2)H^2 - 2\,\frac{\dot{s}}{s}\,H + (D-2)N^2\frac{k}{a^2}\Big], \tag{B.6}$$

$$S_m = \int d^D x\sqrt{-g}\Big(\frac{1}{2N^2}F_{MN}\dot{\Phi}^M\dot{\Phi}^N - \mathcal{F}\Big). \tag{B.7}$$

Here $\{\vec{\Phi}\}$ are the background scalars, which are supposed to be homogeneous in the $(D-1)$-dimensional space. In string theory, they are taken to be the moduli field. We also include $\mathcal{F}$ the free energy density induced by the fluid filling the universe which, on general grounds, depends on everything else N, a and $\vec{\Phi}$. Also, we dress the Ricci curvature R with a function depending on the background scalars $s = s(\vec{\Phi})$. For example when writing down the action in string frame, we have $s = e^{-\frac{2D}{D-2}\phi}$.

The variation of the action S_0 and S_1 against N and a gives

$$\frac{\delta S_g}{\delta N} = \frac{s\,a^{D-1}}{N^2}\frac{(D-1)(D-2)}{2}\Big(H^2 + N^2\frac{k}{a^2}\Big) + (D-1)\frac{a^{D-1}}{N^2}\dot{s}H, \tag{B.8}$$

$$\begin{aligned}\frac{\delta S_g}{\delta a} &= (D-1)(D-2)\frac{s}{N}\Big[\dot{H} - \frac{\dot{N}}{N}H + \frac{1}{2}(D-1)H^2 + (D-3)N^2\frac{k}{a^2}\Big]a^{D-2} \\ &\quad + (D-1)(D-2)\frac{a^{D-2}}{N}\dot{s}H + (D-1)a^{D-2}\frac{d}{dt}\Big(\frac{\dot{s}}{N}\Big)\,;\end{aligned} \tag{B.9}$$

$$\frac{\delta S_m}{\delta N} = -\frac{a^{D-1}}{2N^2}F_{MN}\dot{\Phi}^M\dot{\Phi}^N - a^{D-1}\rho, \tag{B.10}$$

$$\frac{\delta S_m}{\delta a} = (D-1)\,a^{D-2}N\Big(\frac{1}{2N^2}F_{MN}\dot{\Phi}^M\dot{\Phi}^N + P\Big), \tag{B.11}$$

where we defined the energy density and the pressure of the fluid filling the universe

$$\rho = N\frac{\partial\mathcal{F}}{\partial N} + \mathcal{F}\,, \qquad P = -\frac{a}{D-1}\frac{\partial\mathcal{F}}{\partial a} - \mathcal{F}\,. \tag{B.12}$$

Here the definition of ρ and P follows the standard formula in general relativity, that is, we derive the energy momentum tensor $T_{\mu\nu}$ corresponding to the fluid, and then read off the energy and the pressure from $T^\mu_{\ \nu} = \mathrm{diag}(-\rho, P, \ldots, P)^\mu_{\ \nu}$.

For simplicity we put $N = 1$. Then the equations of motion from $\frac{\delta}{\delta N}$, $\frac{\delta}{\delta a}$ and $\frac{\delta}{\delta\Phi^M}$ are respectively

$$\frac{1}{2}(D-1)(D-2)s\Big(H^2 + \frac{k}{a^2}\Big) + (D-1)\dot{s}\,H - \frac{1}{2}F_{MN}\dot{\Phi}^M\dot{\Phi}^N - \rho = 0, \tag{B.13}$$

$$\frac{1}{2}(D-2)\,s\Big[2\dot{H} + (D-1)H^2 + (D-3)\frac{k}{a^2}\Big] + \ddot{s} + (D-2)\dot{s}H + \frac{1}{2}F_{MN}\dot{\Phi}^M\dot{\Phi}^N + P = 0, \tag{B.14}$$

$$\frac{d}{dt}\big(F_{MN}\dot{\Phi}^N\big) - \frac{1}{2}F_{NP,M}\dot{\Phi}^N\dot{\Phi}^P + (D-1)HF_{MN}\dot{\Phi}^N - \frac{1}{2}R\,s_{,M} + \mathcal{F}_{,M} = 0. \tag{B.15}$$

A direct but tedious computation yields the continuity equation

$$\dot{\rho} + (D-1)H(\rho + P) = \dot{\Phi}^M\mathcal{F}_{,M}\,. \tag{B.16}$$

This is just $T^{\mu\nu}_{\ \ ;\nu} = 0$, which explains why the equation is independent of s. The continuity equation can replace Eq.(B.14) when solving for the cosmological evolution.

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Massless D-strings and moduli stabilization in type I cosmology

John Estes,[a,b] Lihui Liu[c] and Hervé Partouche[c]

[a] *Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven, Celestijnenlaan 200D B-3001 Leuven, Belgium*

[b] *Laboratoire de Physique Théorique, Ecole Normale Supérieure,[1] 24 rue Lhomond, F–75231 Paris cedex 05, France*

[c] *Centre de Physique Théorique, Ecole Polytechnique,[2] F–91128 Palaiseau cedex, France*

E-mail: johnalondestes@gmail.com, lihui.liu@cpht.polytechnique.fr, herve.partouche@cpht.polytechnique.fr

Abstract: We consider the cosmological evolution induced by the free energy F of a gas of maximally supersymmetric heterotic strings at finite temperature and weak coupling in dimension $D \geq 4$. We show that F, which plays the role of an effective potential, has minima associated to enhanced gauge symmetries, where all internal moduli can be attracted and dynamically stabilized. Using the fact that the heterotic/type I S-duality remains valid at finite temperature and can be applied at each instant of a quasi-static evolution, we find in the dual type I cosmology that all internal NS-NS and RR moduli in the closed string sector and the Wilson lines in the open string sector can be stabilized. For the special case of $D = 6$, the internal volume modulus remains a flat direction, while the dilaton is stabilized. An essential role is played by light D-string modes wrapping the internal manifold and whose contribution to the free energy cannot be omitted, even when the type I string is at weak coupling. As a result, the order of magnitude of the internal radii expectation values on the type I side is $\sqrt{\lambda_{\rm I}\,\alpha'}$, where $\lambda_{\rm I}$ is the ten-dimensional string coupling. The non-perturbative corrections to the type I free energy can alternatively be described as effects of "thermal E1-instantons", whose worldsheets wrap the compact Euclidean time cycle.

Keywords: Superstrings and Heterotic Strings, String Duality

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[1] Unité mixte du CNRS et de l'Ecole Normale Supérieure associée à l'Université Pierre et Marie Curie (Paris 6), UMR 8549.

[2] Unité mixte du CNRS et de l'Ecole Polytechnique, UMR 7644.

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Contents

1 Introduction

The SO(32) heterotic and type I strings are dual perturbative descriptions of the same underlying theory [1–4]. This is easily observed at the level of the low energy effective actions, which are equivalent after certain field redefinitions. This follows from the fact that short massless supermultiplets have protected masses and that $\mathcal{N}_{10} = 1$ supergravity coupled to ten dimensional super Yang-Mills theory with given gauge group is unique at the two-derivative level. One interesting facet of the equivalence is that in ten dimensions, the heterotic and type I string couplings are inverse to one another, and thus one has the opportunity to uncover strong coupling effects. In dimension $D \geq 7$ ($D \leq 5$), this leads to a strong-weak (weak-weak) duality, while for $D = 6$, string couplings and internal volumes are interchanged [5–8].

In the literature, most of the applications of string dualities have been based on BPS states and therefore restricted to models where supersymmetry is preserved in static universes. In general, extending these ideas to non-supersymmetric cases (see [9, 10] for some examples) and cosmological evolutions is difficult. However, such a project can still be addressed within the context of no-scale models [11–14]. The latter are defined at the classical level by backgrounds associated to vanishing minima of a scalar potential, which admit a flat direction parameterized by the scale of spontaneous supersymmetry breaking. The non-trivial vacuum energy, which arises at the quantum level, backreacts on the flat and originally static universe, and induces a quasi-static time evolution in the background fields [15].

To be specific, start with a dual pair of supersymmetric heterotic and type I models. As follows from the adiabatic argument of [16], one may implement on both sides a spontaneous breaking of supersymmetry, thus giving a new dual pair. For example, if the heterotic theory is in a perturbative regime and the spontaneous breaking at the classical level is compatible with flat Minkowski space, the cosmological evolution induced at the one-loop level can be reinterpreted in the dual type I regime. In this paper, we spontaneously break supersymmetry by considering the models at finite temperature. This can be implemented at the level of the two dimensional CFT by compactifying the Euclidean time on a circle, whose boundary conditions depend on the fermion number [17]. In this case, the one-loop heterotic effective potential discussed above is nothing but the free energy of a perfect gas of supersymmetric strings. Applying the heterotic/type I duality, we find the existence of novel contributions to the type I effective potential coming from light D-strings. Despite being non-perturbative, these corrections have a large impact on the cosmological evolution, as well as on the low energy spectrum of the theory, even at weak type I string coupling.

A second method to spontaneously break supersymmetry is by introducing "geometric" fluxes along internal cycles [18–22]. When the R-symmetry charge associated to the flux is the fermion number, this method is related to the finite temperature case by a double Wick rotation. In this paper, we only explore the thermal breaking for simplicity and clarity, as most of our results have a direct generalization to the second case. In realistic situations, one must include zero temperature spontaneous supersymmetry breaking before switching on finite temperature. In this case, a general picture arises, where the induced cosmology can be divided into different stages. In the Hagedorn era, where the temperature T is close to the string scale $M_{\rm s}$, a phase transition between pre- and post-big bang evolutions takes place. It can be described along the lines of refs. [23, 24] at the level of the two dimensional CFT and is both free of initial singularity and consistent with perturbation theory. As the temperature drops, the cosmology induced by the one-loop effective potential can be trusted until infrared effects become relevant, such as in the cases of radiative breaking or confining gauge groups. For example, in standard GUT scenarios, this defines intermediate eras where the temperature evolves in either of the ranges $M_{\rm s} > T > \Lambda_{\rm GUT}$ or $\Lambda_{\rm GUT} > T > M_{\rm EW}$, where $\Lambda_{\rm GUT}$ and $M_{\rm EW}$ are the GUT and electroweak scales [25–30]. These intermediate eras are connected by a phase transition where the dynamics responsible for the breaking of the GUT group must be taken account. After the electroweak phase transition, the conventional history of the universe follows with the hadronic, leptonic and nucleosynthesis eras...

One feature of the above Hagedorn and intermediate eras is the possibility to stabilize internal moduli [26, 31, 32]. This is an important issue since current observations of the gravitational force place lower limits on scalar masses (see for example [33]). Many approaches address this question by considering compactification spaces where (geometrical or non-geometrical) internal fluxes are switched on at the outset, while preserving some amount of supersymmetry [34–38]. This leads to a partial stabilization since flat directions always persist in such models, at least at the perturbative level. However, we would like to stress that once supersymmetry is broken, flat directions are generically lifted in string theory. This was considered long ago in non-supersymmetric heterotic string backgrounds, such as the SO(16) × SO(16) tachyon free theory toroidally compactified [39, 40]. However, minimization of the moduli-dependent "cosmological constant" generated by loop corrections in such models leads to an unacceptably large vacuum energy at the minima, since supersymmetry is explicitly broken at the string scale. In [41, 42], it was realized that a gas of string modes, which carry both winding and momenta, generate a free energy that enables stabilization of radii moduli. Upon introducing a zero temperature spontaneous breaking of supersymmetry at the string tree level, it was shown in [27–32, 43] that this effect also has a quantum version, with the thermal gas and free energy replaced by virtual strings which induce an effective potential.[1] An advantage of this type of stabilization is that during the intermediate eras, the induced masses are not constant. Instead, they follow the time-evolution of the temperature $T(t)$ and supersymmetry breaking modulus $M(t)$, which drop proportionally. It is only after the electroweak phase transition that $M(t)$ is stabilized and that the induced moduli masses become constant. As a result, the energy of the moduli with time-dependent masses is diluted during the intermediate eras, and the cosmological moduli problem [47–49][2] is avoided. Moreover, the decrease of $M(t) \propto T(t)$ during the intermediate eras gives a dynamical explanation of the hierarchy between the supersymmetry breaking scale and the string scale, $M \ll M_s$.

This above dynamical moduli stabilization relies on the existence of perturbative states in the string spectrum, whose masses are determined by the expectation value of the moduli and vanish at the stabilization points. For instance, in toroidal or orbifold compactifications of the heterotic string, if the radius R_i of some factorized internal circle is not participating in the spontaneous breaking of supersymmetry, it can be attracted to the self dual point $R_i = 1$ associated to an enhanced SU(2) level one Kac Moody algebra. Another simple example can be realized in type II superstring, when the internal circle is used to spontaneously break the supersymmetries generated by the right-moving sector via the Scherck Schwarz mechanism. In this case, R_i can be stabilized at the fermionic point $R_i = 1/\sqrt{2}$ corresponding to a Kac Moody level two SU(2) extension [23]. However, since this type II setup is intrinsically left/right asymmetric, it cannot be extended to orientifold models in a straightforward way. Thus, the purpose of the present work is to infer how the internal moduli in type I no-scale models are stabilized by using our knowledge of the dual heterotic picture. As said before, we consider only thermal effects, as this is sufficient to uncover the mechanism. More specifically, using heterotic/type I duality at finite temperature, we infer the existence of non-pertrubative contributions to the thermal free energy of type I superstrings. These contributions are due to light, or even massless, D-strings which wrap the internal cycles and participate to the dynamical stabilization of all the internal moduli, including those in the RR sector and the Wilson lines.

[1]In refs. [44–46], the effect of the Coleman-Weinberg effective potential is explicitly subtracted in order to isolate the backreaction on the moduli arising from particle production near extra massless species points. To be substantial, this mechanism supposes the moduli already have non-trivial motions at tree level. Since the no-scale models are based on classical static backgrounds, the moduli velocities occur as backreactions of the one-loop effective potential and particle production is higher order in perturbation theory.

[2]A simplified statement of this problem is that the energy of scalars with constant masses dilutes slower than the thermal energy of radiation, and so heavy scalars tend to dominate at late times, which can cause problems for nucleosynthesis. This may be fixed by requiring the heavy scalars to be unstable so that their fluctuations eventually decay, thereby reheating space-time. However, the reheating process creates extra entropy and one can run into problems with baryogensis.

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We derive in section 2 the free energy of a gas of weakly coupled perturbative states in type I superstring, in the simple case where the internal space is a factorized torus. We describe the induced cosmological evolution and find the radii moduli are flat directions of the thermal potential. In section 3, using the dual heterotic model at weak coupling, we correct this naive analysis by taking into account contributions of non-perturbative states to the free energy. In particular, D-strings modes are found to be light when the radii are in a neighborhood of $\sqrt{\lambda_{\rm I}}$, where $\lambda_{\rm I} \gg 1$ is the ten dimensional type I string coupling. They produce local minima of the thermal potential which are responsible for the stabilization of the radii at $\sqrt{\lambda_{\rm I}}$. In type I, this dynamical effect occurs at strong (weak) coupling when $D \geq 7$ ($D \leq 6$). However, since the BPS masses of the light D1-branes are protected by supersymmetry, our results are also valid at small string coupling for $D \geq 7$. In section 4, we reexamine the form of the corrections to the free energy along the lines of [50, 51], and interpret the non-perturbative contributions as arising from "thermal E1-instantons". What is meant by this is that the Euclidean worldsheets of the D1-branes wrap the Euclidean time circle. In section 5, we generalize our results: The one-loop heterotic free energy is computed, with all of the internal moduli taken into account. We find that at certain points in moduli space, all scalars, except the dilaton, may be stabilized for $D \geq 4$.[3] On the dual type I side, the non-perturbative effects induce a stabilization of the internal NS-NS and RR moduli in the closed string sector, and the Wilson lines in the open string sector. For the special case of $D = 6$, the internal volume modulus remains a flat direction, while the dilaton is stabilized at a small value. In section 6, we give explicit examples of loci in moduli space where only the flat direction of the dilaton survives. Section 7 is devoted to our conclusions and perspectives.

2 Naive perturbative type I thermal cosmology

In this section, we derive the cosmology induced by thermal effects in the purely perturbative type I superstring theory toroidally compactified down to $D \geq 3$ dimensions. We shall see in the next section how light solitonic states correct this picture in a drastic way. In the following, quantities are denoted in the type I context with subscripts I and, throughout this paper, "hatted" ("un-hatted") ones are referring to the string (Einstein) frame. Finite temperature $\hat{T}_{\rm I}$ is implemented by considering an Euclidean time of period $\hat{\beta}_{\rm I} = 2\pi R_{\rm I0} = 1/\hat{T}_{\rm I}$, and coupling the associated $S^1(R_{\rm I0})$ lattice of zero modes to the fermion number. We restrict for the moment our study to the case of a factorized internal space $\prod_{i=D}^{9} S^1(R_{{\rm I}i})$ and analyze the dynamics of the radii $R_{{\rm I}i}$.

Working in a perturbative regime, there are four contributions to the Euclidean one-loop partition function needed to express the free energy density, namely the torus, Klein-bottle, annulus and Möbius strip vacuum-to-vacuum amplitudes $\mathcal{T}$, $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$. In units

[3] Additionally, for $D \geq 5$ the dilaton approaches a constant finite value at late times and the cosmological evolution is radiation dominated. For $D = 4$, the dilaton decreases logarithmically with cosmological time and the coherent motion of all moduli is such that the metric evolution is that of a radiation dominated universe, $H^2 \propto 1/a^4$. However, non-perturbative effects from NS5 or D5-branes in the heterotic or type I theories should be taken into account in four dimensions and may play a role in stabilizing the dilaton.

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where $\alpha' = 1$, a little work yields (see the appendix),

$$\mathcal{T} = \frac{\hat\beta_{\rm I}\hat V_{\rm I}}{\hat\beta_{\rm I}^D}\left\{ s_0^2\, c_D + \sum_{\substack{A\ge 0,\, \bar A\ge 0,\, \vec m,\, \vec n \\ A-\bar A=\vec m\cdot\vec n \\ (A,\vec m,\vec n)\neq(0,\vec 0,\vec 0)}} s_A s_{\bar A}\, G\left(2\pi R_{\rm I0}\left[4A + \sum_{i=D}^{9}\left(\frac{m_i}{R_{{\rm I}i}} - n^i R_{{\rm I}i}\right)^2\right]^{\frac12}\right)\right\}, \tag{2.1}$$

where $\hat V_{\rm I}$ is the regularized volume of the $(D-1)$-dimensional space, c_D is Stefan's constant for radiation in dimension D and the function G is defined in terms of a modified Bessel function of the second kind, $K_{\frac{D}{2}}(x)$:

$$c_D = \frac{\Gamma\left(\frac{D}{2}\right)}{\pi^{\frac{D}{2}}}\sum_{\tilde k_0}\frac{1}{|2\tilde k_0+1|^D}\,, \qquad G(x) = 2\sum_{\tilde k_0}\left(\frac{x}{2\pi|2\tilde k_0+1|}\right)^{\frac{D}{2}} K_{\frac{D}{2}}\left(x\,|2\tilde k_0+1|\right). \tag{2.2}$$

The integer s_A ($s_{\bar A}$) counts the degeneracy at oscillator level A ($\bar A$) on the left (right)-moving side of the worldsheet, while m_i (n^i) labels the momentum (winding) number along the i-th cycle of the internal torus.[4] In (2.1), the first term in the braces is the contribution of the massless modes, with quantum numbers $(A,\vec m,\vec n) = (0,\vec 0,\vec 0)$ and associated to the $\mathcal{N}_{10} = 1$ supergravity multiplet in ten dimensions. The Klein-bottle contribution $\mathcal{K}$ vanishes. The annulus plus Möbius amplitude takes in a similar way the form

$$\mathcal{A}+\mathcal{M} = \frac{\hat\beta_{\rm I}\hat V_{\rm I}}{\hat\beta_{\rm I}^D}\left\{\frac{N^2-N}{2}\, s_0\, c_D + \sum_{\substack{A\ge 0,\, \vec m \\ (A,\vec m)\neq(0,\vec 0)}}\frac{N^2-(-1)^A N}{2}\, s_A\, G\left(2\pi R_{\rm I0}\left[A+\sum_{i=D}^{9}\left(\frac{m_i}{R_{{\rm I}i}}\right)^2\right]^{\frac12}\right)\right\}, \tag{2.3}$$

where $N = 32$ and the first term is associated to the $\mathcal{N}_{10} = 1$ SO(32) super-vector multiplet in ten dimensions. The partition function is given by the sum $Z_{\rm I} = \mathcal{T}+\mathcal{K}+\mathcal{A}+\mathcal{M}$. At high temperatures, it becomes ill-defined. Examining $\mathcal{T}$, one finds that winding modes along the Euclidean time circle become tachyonic when $R_{\rm I0} < R_{\rm IH}$, where $R_{\rm IH} = \sqrt{2}$ is the Hagedorn radius. This divergence of $Z_{\rm I}$ is not a sickness of the theory, but rather the signal of a phase transition [52–55]. From now on, we restrict ourselves to temperatures below $\hat T_{\rm IH} \equiv 1/(2\pi R_{\rm IH})$.

The free energy density is defined in terms of the partition function as $\hat{\mathcal{F}}_{\rm I} = -Z_{\rm I}/(\hat\beta_{\rm I}\hat V_{\rm I})$. It is expressed in terms of the G-function, whose arguments are the ratios of the spectrum masses to the temperature. Since

$$G(x) = c_D - \frac{c_{D-2}}{4\pi}\,x^2 + \mathcal{O}(x^4) \;\text{ when }\; x\simeq 0\,, \quad G(x) \sim \left(\frac{x}{2\pi}\right)^{\frac{D-1}{2}} e^{-x} \;\text{ when }\; x\gg 1, \tag{2.4}$$

the dominant contribution at low temperature (compared to the string scale) arises from the first terms of (2.1) and (2.3) and corresponds to the free energy density of thermal radiation,

$$\hat{\mathcal{F}}_{\rm I} = -\left(s_0^2 + \frac{N^2-N}{2}\, s_0\right) c_D\, \hat T_{\rm I}^D + \cdots\,. \tag{2.5}$$

[4] Note that the condition $A - \bar A = \vec m\cdot\vec n$ provides the level matching.

However, if some $R_{\mathrm{I}i}$ is large (small) enough, $R_{\mathrm{I}i} > 2\pi R_{\mathrm{I}0}$ ($R_{\mathrm{I}i} < 1/(2\pi R_{\mathrm{I}0})$), pure Kaluza-Klein (winding) modes yield additional terms of the same order. The contributions associated to the remaining states are exponentially suppressed.

It is straightforward to apply the techniques introduced in [31, 32] for closed strings to study the backreaction of the type I free energy on the originally static background. For arbitrary initial conditions at the exit of the Hagedorn era, one finds that the system is attracted to a radiation dominated evolution, where all internal radii and the dilaton are frozen at non-specific values depending on the initial data. Quantitatively, the final constant values of the $R_{\mathrm{I}i}$'s sit in the range

$$\frac{1}{2\pi R_{\mathrm{I}0}} < R_{\mathrm{I}i} < 2\pi R_{\mathrm{I}0}\ , \quad i = D, \dots, 9, \tag{2.6}$$

where $R_{\mathrm{I}0}$ is increasing with time, corresponding to an expanding and cooling universe. Actually, if at some time t a radius $R_{\mathrm{I}j}$ is outside this range, we find $R_{\mathrm{I}j}(t)$ and $R_{\mathrm{I}0}(t)$ always evolve so that the condition (2.6) is finally satisfied, after which the evolution of $R_{\mathrm{I}j}$ comes to a halt. This may be seen by examining the force on the modulus $\mu_j = \ln(2\pi R_{\mathrm{I}0}/R_{\mathrm{I}j})$ (or $\ln(2\pi R_{\mathrm{I}0} R_{\mathrm{I}j})$) [31, 32].

A difference compared to the type II and heterotic string cases, is that the open string sector is not invariant under T-duality, $R_{\mathrm{I}i} \to 1/R_{\mathrm{I}i}$ (for any i), due to a lack of winding quantum numbers in the open sector. For instance, for arbitrary $R_{\mathrm{I}j}$ (for a given j), while the other radii satisfy (2.6), the effective potential for $R_{\mathrm{I}j}$, which is exactly the free energy density, simplifies to

$$\begin{aligned}
\hat{\mathcal{F}}_{\mathrm{I}} &= -\hat{T}_{\mathrm{I}}^D \left\{ \left(s_0^2 + \frac{N^2-N}{2}\, s_0 \right) \left[c_D + \sum_{m_j \neq 0} G\left(2\pi R_{\mathrm{I}0} \frac{|m_j|}{R_{\mathrm{I}j}} \right) \right] + \mathcal{O}(e^{-2\pi R_{\mathrm{I}0}}) \right\}, && 2\pi R_{\mathrm{I}0} < R_{\mathrm{I}j} \\
\hat{\mathcal{F}}_{\mathrm{I}} &= -\hat{T}_{\mathrm{I}}^D \left\{ \left(s_0^2 + \frac{N^2-N}{2}\, s_0 \right) c_D + \mathcal{O}(e^{-2\pi R_{\mathrm{I}0}}) \right\}, && \frac{1}{2\pi R_{\mathrm{I}0}} < R_{\mathrm{I}j} < 2\pi R_{\mathrm{I}0} \\
\hat{\mathcal{F}}_{\mathrm{I}} &= -\hat{T}_{\mathrm{I}}^D \left\{ \frac{N^2-N}{2}\, s_0\, c_D + s_0^2 \left[c_D + \sum_{n_j \neq 0} G\left(2\pi R_{\mathrm{I}0} |n_j| R_{\mathrm{I}j} \right) \right] + \mathcal{O}(e^{-2\pi R_{\mathrm{I}0}}) \right\}, && R_{\mathrm{I}j} < \frac{1}{2\pi R_{\mathrm{I}0}}
\end{aligned} \tag{2.7}$$

and is shown in figure 1, in Einstein frame. When $R_{\mathrm{I}j} < 1$, the theory is actually better understood in the T-dual type I' picture obtained by inverting $R_{\mathrm{I}j}$. More importantly, there is no local minimum of the free energy density where $R_{\mathrm{I}j}$ (as well as the $R_{\mathrm{I}i}$'s) can be attracted and stabilized. This is contrary to the heterotic case, where enhanced symmetry points exist and imply a local increase of the number of massless states. However, we shall find that the above purely perturbative analysis is missing important contributions from massless solitons.

3 Heterotic/type I cosmological duality

Given that heterotic and type I theories at zero temperature are S-dual in ten dimensions, it is a simple but non-trivial fact that they remain S-dual at finite temperature. Technically, the backgrounds used to analyze the thermal ensembles are freely acting orbifolds, obtained

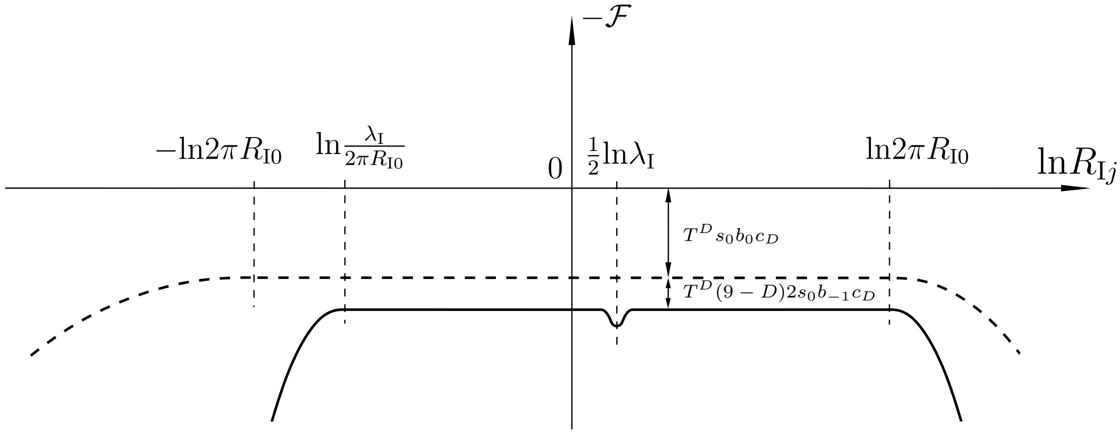

Figure 1. Thermal effective potential (in Einstein frame) for $R_{\mathrm{I}j}$, when all other internal radii satisfy (2.6). The dashed curve takes only into account the perturbative type I states. The solid one is obtained by heterotic/type I S-duality and receives corrections from light D-string modes.

by modding out with $(-1)^F\delta_0$, where δ_0 is an order-two shift along the Euclidean time circle and F is the fermion number. Using the "adiabatic argument" of [16], after such a free action, the two theories remain dual. Since the cosmological evolutions we study are quasi-static, it is valid to apply at each time an S-duality transformation on the heterotic side, in order to derive non-perturbative contributions to the type I free energy and its resulting backreaction.

S-dual SO(32) heterotic string. Let us apply this point of view to the type I background considered in section 2. The dual theory is the SO(32) heterotic string compactified on $\prod_{i=D}^{9} S^1(R_{\mathrm{h}i})$, where we use the subscript h to denote heterotic quantities. As in the type I case, the partition function is only well defined when the temperature $\hat{T}_{\mathrm{h}} = 1/\hat{\beta}_{\mathrm{h}} = 1/(2\pi R_{\mathrm{h}0})$ is below the heterotic Hagedorn temperature, i.e.$R_{\mathrm{h}0} > R_{\mathrm{hH}} \equiv (1+\sqrt{2})/\sqrt{2}$. As shown in the appendix, the heterotic partition function can be brought into a form divided in three parts as follows:

$$Z_{\mathrm{h}} = \frac{\hat{\beta}_{\mathrm{h}}\hat{V}_{\mathrm{h}}}{\hat{\beta}_{\mathrm{h}}^D}\Bigg\{s_0 b_0\, c_D + \sum_{i=D}^{9} 2 s_0 b_{-1}\, G\bigg(2\pi R_{\mathrm{h}0}\bigg|\frac{1}{R_{\mathrm{h}i}} - R_{\mathrm{h}i}\bigg|\bigg) \\ + \sum_{\substack{A\geq 0,\, \bar{A}\geq -1,\, \vec{m},\, \vec{n} \\ A-\bar{A}=\vec{m}\cdot\vec{n} \\ (A,\vec{m},\vec{n})\neq(0,\epsilon\vec{e}_i,\epsilon\vec{e}_i), \\ \forall i, \forall\epsilon=-1,0,1}} s_A b_{\bar{A}}\, G\Bigg(2\pi R_{\mathrm{h}0}\Bigg[4A + \sum_{j=D}^{9}\bigg(\frac{m_j}{R_{\mathrm{h}j}} - n^j R_{\mathrm{h}j}\bigg)^2\Bigg]^{\frac{1}{2}}\Bigg)\Bigg\}, \tag{3.1}$$

where the degeneracy $b_{\bar{A}}$ of the right-moving bosonic string oscillator modes is defined from level -1. The first contribution in Z_{h} is associated to the massless states labeled by $(A,\vec{m},\vec{n}) = (0,\vec{0},\vec{0})$. They arise from the $\mathcal{N}_{10} = 1$ supergravity and SO(32) super-vector multiplets in ten dimensions. The second contribution comes from modes whose masses can vanish at particular values of the internal radii. For each i, these states are labeled

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as $(A,\vec{m},\vec{n}) = (0,\epsilon\,\vec{e}_i,\epsilon\,\vec{e}_i)$, where $\epsilon = \pm 1$ and $\vec{e}_i$ is the unit vector in the direction i. They are massless at the self-dual point $R_{\mathrm{h}i} = 1$, where they enhance the gauge symmetry, $\mathrm{U}(1) \to \mathrm{SU}(2)$. The last line in (3.1) arises from the states which are never massless. It becomes substantial when Kaluza-Klein (winding) states become light, in the regime where some $R_{\mathrm{h}i}$'s are large (small) compared to $2\pi R_{\mathrm{h}0}$ $(1/(2\pi R_{\mathrm{h}0}))$. All other modes, being always super heavy as compared to the temperature scale, yield to exponentially suppressed contributions (see eq. (2.4)).

Duality map. In ten dimensions, the heterotic/type I S-duality identifies the Einstein frame metrics and inverts the string couplings, $\lambda_{\mathrm{h}} = e^{\phi_{\mathrm{h}}^{(10)}} = e^{-\phi_{\mathrm{I}}^{(10)}} = 1/\lambda_{\mathrm{I}}$ [1–4]. In lower dimension D, these relations translate into a dictionary between the Einstein frame metrics, the internal radii and dilatons [5–8]:

$$
\begin{aligned}
ds^2_{\mathrm{h}(D)} &= ds^2_{\mathrm{I}(D)} \\
R_{\mathrm{h}i} &= \frac{R_{\mathrm{I}i}}{\sqrt{\lambda_{\mathrm{I}}}} \equiv R_{\mathrm{I}i}\,\frac{e^{-\frac{1}{2}\phi_{\mathrm{I}}^{(D)}}}{\left(\prod_{j=D}^{9} 2\pi R_{\mathrm{I}j}\right)^{1/4}}\,, \qquad i = 0 \text{ or } D,\dots,9, \\
\phi_{\mathrm{h}}^{(D)} &= -\frac{D-6}{4}\,\phi_{\mathrm{I}}^{(D)} - \frac{D-2}{8}\sum_{i=D}^{9}\ln\left(2\pi R_{\mathrm{I}i}\right),
\end{aligned}
\tag{3.2}
$$

where the D-dimensional dilatons are defined as $\phi_{\mathrm{h,I}}^{(D)} = \phi_{\mathrm{h,I}}^{(10)} - \frac{1}{2}\sum_{i=D}^{9}\ln(2\pi R_{\mathrm{h,I}i})$. Note that the Euclidean radii $R_{\mathrm{I}0}$ and $R_{\mathrm{h}0}$ are included in the above relations. The inverse maps, which relate the type I fields in terms of heterotic quantities, are obtained by exchanging the subscripts $\mathrm{h} \leftrightarrow \mathrm{I}$.

We consider non-trivial evolutions for the Einstein frame metric, dilaton and internal radii moduli. It is easily checked that the tree level heterotic and type I actions match, under the S-duality transformation (3.2) (i.e.$S_{\mathrm{h}}^{\mathrm{tree}} = S_{\mathrm{I}}^{\mathrm{tree}}$). The one-loop finite temperature effective potentials were computed using Euclidean backgrounds with laps functions $R_{\mathrm{h,I}0}$ in the string frames. For the Lorentzian Einstein frame metric,

$$
\begin{aligned}
ds^2_{(D)} &= \frac{1}{(2\pi)^2}\left[-\beta(x^0)^2 dx^{0\,2} + a(x^0)^2\left(dx^{1\,2} + \cdots + dx^{D-1\,2}\right)\right] \\
\beta &= 2\pi R_{\mathrm{h,I}0}\, e^{-\frac{2}{D-2}\phi_{h,\mathrm{I}}^{(D)}}\,, \qquad a^{D-1} = \hat{V}_{h,\mathrm{I}}\, e^{-\frac{2(D-1)}{D-2}\phi_{h,\mathrm{I}}^{(D)}}\,,
\end{aligned}
\tag{3.3}
$$

the corresponding first order correction to the tree level action $S_{\mathrm{h,I}}^{\mathrm{tree}}$ is given by

$$
S_{\mathrm{h,I}}^{\text{1-loop}} = -\int d^D x\sqrt{-g^{(D)}}\,\mathcal{F}_{\mathrm{h,I}} \qquad \text{where} \qquad \mathcal{F}_{\mathrm{h,I}} = -\frac{Z_{\mathrm{h,I}}}{\beta a^{D-1}}. \tag{3.4}
$$

Note that we do not distinguish between the heterotic and type I inverse temperature β and scale factor a in (3.3), as they are measured in Einstein frame and are identified under the duality map (3.2). To be exactly equivalent, the effective actions should be corrected to all orders in perturbation theory and include non-perturbative effects as well. In the following, we will consider the heterotic point of view at weak coupling, $e^{\phi_{\mathrm{h}}^{(D)}} \ll 1$, restrict our computations at the one-loop order, and deduce the type I behavior in the dual regime.

Dual type I cosmological evolution. To start, we apply the duality map to (3.1) and note that the first term exactly matches the sum of the first contributions in (2.1) and (2.3). This follows from the equality $b_0 = s_0 + (N^2 - N)/2$ and is due to the fact that the supergravity and SO(32) super-vector multiplets are short, with protected vanishing masses. Next, we concentrate on the interpretation and cosmological implications of the remaining terms in (3.1). In the weakly coupled heterotic string, $e^{\phi_{\rm h}^{(D)}} \ll 1$, the time evolution of the universe for arbitrary initial conditions at the exit of the Hagedorn era can be analyzed along the lines of refs. [31, 32]. We first summarize the results here and then derive the dual type I cosmological behavior:

- When all radii satisfy $|R_{{\rm h}i} - 1/R_{{\rm h}i}| < 1/(2\pi R_{\rm h0})$, $i = D, \dots, 9$, the heterotic free energy density derived from (3.1) takes the form:

$$\mathcal{F}_{\rm h} = -T^D \left\{ s_0 b_0\, c_D + \sum_{i=D}^{9} 2 s_0 b_{-1}\, G\left(2\pi R_{\rm h0}\left|\frac{1}{R_{{\rm h}i}} - R_{{\rm h}i}\right|\right) + \mathcal{O}(e^{-2\pi R_{\rm h0}}) \right\}. \tag{3.5}$$

  Thanks to the properties (2.4), the states with quantum numbers $(A, \vec{m}, \vec{n}) = (0, \epsilon\vec{e}_i, \epsilon\vec{e}_i)$ are responsible for the existence of a local minimum of $\mathcal{F}_{\rm h}$ at $R_{{\rm h}D} = \cdots = R_{\rm h9} = 1$. The internal radii can be attracted and stabilized at this $\mathrm{SU}(2)^{10-D}$ enhanced symmetry point. Moreover, for $D \geq 5$ the string coupling $e^{\phi_{\rm h}^{(D)}}$ (and thus $\lambda_{\rm h}$) freezes to some constant value $e^{\phi_{\rm h0}^{(D)}}$ determined by the initial conditions. For $D = 4$, the dilaton $\phi_{\rm h}^{(4)}$ does not converge to a constant but instead decreases logarithmically with cosmological time. We show this in section 5 in a general context where we take into account all internal moduli. The rest of this section is valid for $D \geq 5$, while for $D = 4$ one has to keep in mind the late time evolution of $\phi_{\rm h}^{(4)}$.

  Applying the duality map (3.2), the ratios of the masses of the above winding-momentum states to the temperature become:

$$\frac{\hat{M}_{{\rm h}i}}{\hat{T}_{\rm h}} \equiv 2\pi R_{\rm h0}\left|R_{{\rm h}i} - \frac{1}{R_{{\rm h}i}}\right| = 2\pi R_{\rm I0}\left|\frac{R_{{\rm I}i}}{\lambda_{\rm I}} - \frac{1}{R_{{\rm I}i}}\right| \equiv \frac{\hat{M}_{{\rm I}i}}{\hat{T}_{{\rm I}i}}\,. \tag{3.6}$$

  From the type I point of view, the corresponding BPS states have a natural interpretation as D (or anti-D)-strings wrapped once along the circles $S^1(R_{{\rm I}i})$, with one unit of momentum. The heterotic cosmology translates into the type I context as follows. Whenever the type I radii start out in the dual range $|R_{{\rm I}i}/\lambda_{\rm I} - 1/R_{{\rm I}i}| < 1/(2\pi R_{\rm I0})$, the light D-string modes can stabilize them at the point

$$R_{{\rm I}i} = \sqrt{\lambda_{\rm I0}}\,, \qquad i = D, \dots, 9, \tag{3.7}$$

  where $\lambda_{\rm I0} = 1/\lambda_{\rm h0} \gg 1$ is the late time constant value of the string coupling in ten dimensions. This implies the open string cosmology is well understood in type I, rather than in the T-dual picture in type I'. At each time, the width of the symmetric well of the potential for $\ln R_{{\rm I}i}$ is $\sqrt{\lambda_{\rm I}}/(2\pi R_{\rm I0})$ (see figure 1). In total, if we denote by $\phi_{\rm I0}^{(D)}$ the asymptotic value of the type I dilaton in D dimensions and use the inverse relations (3.2), the moduli are found to converge as follows,

$$e^{\phi_{\rm I}^{(D)}(t)} \longrightarrow e^{\phi_{\rm I0}^{(D)}} \equiv \frac{e^{-\frac{D-6}{4}\phi_{\rm h0}^{(D)}}}{(2\pi)^{\frac{(10-D)(D-2)}{8}}}\,,\;\; R_{{\rm I}i}(t) \longrightarrow e^{\frac{2}{D-6}\phi_{\rm I0}^{(D)}}\,(2\pi)^{\frac{10-D}{D-6}} = \frac{1}{e^{\frac{1}{2}\phi_{\rm h0}^{(D)}}\,(2\pi)^{\frac{10-D}{4}}}, \tag{3.8}$$

while the temperature and scale factor asymptotic behaviors are those of a radiation dominated era, $T^{-1}(t) \sim a(t) \sim t^{2/D}$, where t is the cosmological time. Some remarks are in order:

- ◇ For $D > 6$, (3.8) shows that the type I cosmology is at strong coupling. In this regime, solitons are generically light and the need to include their effects in the low energy effective action is not surprising.
- ◇ For $D = 6$, the asymptotic values of the moduli are $e^{\phi_{\rm I0}^{(6)}} = 1/(2\pi)^2$ and $R_{{\rm I}i}(t) \to e^{-\frac{1}{2}\phi_{\rm h0}^{(6)}}/(2\pi)$. The type I picture is perturbative.
- ◇ For $D < 6$, the type I cosmological evolution is at weak coupling. However, we observe the necessity to take into account the contributions arising from solitons which are light, when we sit in the neighborhood of the enhanced symmetry points.

In summary, for $D \neq 6$ in type I, the internal radii are stabilized while the dilaton $\phi_{\rm I}^{(D)}$ freezes somewhere along its flat direction. On the contrary, for $D = 6$, the dilaton is stabilized, all complex structures $R_{{\rm I}i}/R_{{\rm I}j}$ are also stabilized at one, while the internal space volume $\prod_{i=D}^{9}(2\pi R_{{\rm I}i})$ freezes along a flat direction. This is not a surprise, since in $D = 6$ the heterotic/type I duality exchanges internal volumes and string couplings: $\prod_{i=D}^{9}(2\pi R_{{\rm h,I}i}) \leftrightarrow 1/e^{2\phi_{\rm I,h}^{(6)}}$.

- If at some epoch one of the heterotic internal radii satisfies $R_{{\rm h}j} > 2\pi R_{\rm h0}$, while the $9-D$ remaining ones are stabilized, $R_{{\rm h}i} = 1$ for $i \neq j$, the free energy density deduced from (3.1) becomes

$$\mathcal{F}_{\rm h} = -T^D\left(s_0 b_0 + (9-D) 2 s_0 b_{-1}\right)\left[c_D + \sum_{m_j \neq 0} G\left(2\pi R_{\rm h0}\frac{|m_j|}{R_{{\rm h}j}}\right)\right] + \mathcal{O}(e^{-2\pi R_{\rm h0}}). \quad (3.9)$$

We see that in addition to the massless supergravity and SO(32) super-vector multiplets, there are also contributions coming from their Kaluza-Klein descendants, which are light since $R_{{\rm h}j}$ is large. Applying the duality rules and comparing to the perturbative type I result in the first line of (2.7), we observe a match up to an additional contribution $(9-D)\, 2 s_0 b_{-1}$ to the overall numerical coefficient. This discrepancy arises from the extra massless D (or anti-D)-strings responsible for the stabilization of the $R_{{\rm I}i}$'s at $\sqrt{\lambda_{\rm I}}$. Therefore, the main difference with the pure perturbative analysis is that the plateau of the effective potential is lowered and that the slope for $R_{{\rm I}j} > 2\pi R_{\rm I0}$ is steeper (see figure 1). The cosmological evolution is however similar to the one discussed below (2.6). As their heterotic counterparts [31, 32], $R_{{\rm I}j}(t)$ and $R_{\rm I0}(t)$ evolve such that the regime where $R_{{\rm I}j}(t) < 2\pi R_{\rm I0}(t)$ is reached. After that, $R_{{\rm I}j}$ freezes along its plateau or is stabilized at $\sqrt{\lambda_{\rm I}}$ as explained before.

- In a similar way, if a heterotic radius satisfies $R_{{\rm h}j} < 1/(2\pi R_{\rm h0})$, while the others are stabilized at their self-dual points, $R_{{\rm h}i} = 1$ for $i \neq j$, we have

$$\mathcal{F}_{\rm h} = -T^D\left(s_0 b_0 + (9-D)\, 2 s_0 b_{-1}\right)\left[c_D + \sum_{n_j \neq 0} G\left(2\pi R_{\rm h0}|n_j| R_{{\rm h}j}\right)\right] + \mathcal{O}(e^{-2\pi R_{\rm h0}}). \quad (3.10)$$

In this case, substantial contributions arise from the winding modes along $S^1(R_{\mathrm{h}j})$, which are light since $R_{\mathrm{h}j}$ is small enough. Their effect is to attract $R_{\mathrm{h}j}(t)$ to values larger than $1/(2\pi R_{\mathrm{h}0}(t))$ [31, 32]. Applying the S-duality rules to translate this statement in the type I context, we find that if $R_{\mathrm{I}j} < \lambda_{\mathrm{I}}/(2\pi R_{\mathrm{h}0})$ at some time, the evolution of these moduli implies we end in a regime where $\lambda_{\mathrm{I}}/(2\pi R_{\mathrm{h}0}) < R_{\mathrm{I}j}$, after which the internal modulus freezes or is stabilized at $\sqrt{\lambda_{\mathrm{I}}}$. Noting that the argument of the G-function in (3.10) becomes

$$\frac{\hat{M}_{\mathrm{h}j}}{\hat{T}_{\mathrm{h}}} \equiv 2\pi R_{\mathrm{h}0}\,|n^j| R_{\mathrm{h}j} = 2\pi R_{\mathrm{I}0}\,|n^j| \frac{R_{\mathrm{I}j}}{\lambda_{\mathrm{I}}} \equiv \frac{\hat{M}_{\mathrm{I}j}}{\hat{T}_{\mathrm{I}}}, \tag{3.11}$$

we conclude that the above mechanism is due to two sets of towers of D-string winding modes along $S^1(R_{\mathrm{I}j})$. The first one contains "solitonic descendants" of the perturbative massless supergravity and SO(32) super-vector multiplets. The second set is associated to the descendants of the D (or anti-D)-strings responsible for the stabilization of the $(9-D)$ internal radii $R_{\mathrm{I}i}$ at $\sqrt{\lambda_{\mathrm{I}}}$. The net result of these non-perturbative light states is to render the type I free energy explicitly invariant under the "non-perturbative T-duality" $R_{\mathrm{I}j} \to \lambda_{\mathrm{I}}/R_{\mathrm{I}j}$ (see figure 1).[5]

Comments. To conclude this section, we would like to make some remarks. We first observe that under the duality map (3.2), the Hagedorn radii do not match. We thus infer from the perturbative heterotic side a new value of the Hagedorn radius in type I, when λ_{I} is large:

$$R_{\mathrm{IH}} = \begin{cases} \sqrt{2} & \text{for } \lambda_{\mathrm{I}} \ll 1 \\ \sqrt{\lambda_{\mathrm{I}}}\,\frac{1+\sqrt{2}}{\sqrt{2}} & \text{for } \lambda_{\mathrm{I}} \gg 1 \end{cases}. \tag{3.12}$$

From a cosmological point of view, R_{IH} in the regime $\lambda_{\mathrm{I}}(t) \gg 1$ is thus a time-dependent scale. Note that this non-perturbative expression for R_{IH} obtained once D-strings are taken into account can be relevant even at weak coupling, $e^{\phi_{\mathrm{I}}^{(D)}} \ll 1$. This is for instance the case for $D \leq 6$, when $\sqrt{\lambda_{\mathrm{I}}}$ and the $R_{\mathrm{I}i}$'s reach the asymptotic value $\sqrt{\lambda_{\mathrm{I}0}} \gg 1$.

For $D \geq 7$, the stabilization of the internal type I radii at $\sqrt{\lambda_{\mathrm{I}0}} \gg 1$ occurs at strong coupling, $e^{\phi_{\mathrm{I}}^{(D)}} \gg 1$. However, the D-string states responsible for this effect are BPS, so that their masses are protected by supersymmetry. Thus, these modes remain massless for arbitrary λ_{I}, when $R_{\mathrm{I}i} = \sqrt{\lambda_{\mathrm{I}}}$. It follows that the type I free energy density can easily be determined when $\lambda_{\mathrm{I}} \ll 1$ and $R_{\mathrm{I}i} \simeq \sqrt{\lambda_{\mathrm{I}}}$. It is actually given by (3.5), once translated in terms of dual type I variables. The justification of this statement is based on the following facts. In this regime, the string coupling is weak, $e^{\phi_{\mathrm{I}}^{(D)}} \ll 1$, and the contribution of the perturbative part of the spectrum is that of a perfect gas. Moreover, the contribution of the light solitons is of identical form, since SU(2)'s (gauge) symmetries transform them into the perturbative modes in the Cartan subalgebras. We conclude that the mechanism of stabilization of the internal type I radii remains valid at weak coupling $e^{\phi_{\mathrm{I}}^{(D)}} \ll 1$. Since

[5] Since at late times $\lambda_{\mathrm{I}}(t) \to \lambda_{\mathrm{I}0}$ and $R_{\mathrm{I}0}(t) \to +\infty$, the left-boundary of the plateau of the effective potential of $\ln R_{\mathrm{I}j}$ ends by being negative. This means that $R_{\mathrm{I}j}$ may freeze at some value below one. In such a case, a T-duality $R_{\mathrm{I}j} \to 1/R_{\mathrm{I}j}$ to a type I' description is more suitable. In general, a mixed type I / type I' theory may be obtained, in order to keep all internal radii larger than one.

this yields $R_{\rm Ii} = \sqrt{\lambda_{\rm I0}} \ll 1$, the model is better described in the T-dual type I' picture. However, the dynamics in the intermediate regime $e^{\phi_{\rm I}^{(D)}} \simeq 1$ for $D \geq 7$ (or $e^{\phi_{\rm I}^{(D)}} \not\ll 1$ for $D \leq 6$) cannot be inferred from these arguments.

Finally, for $D \leq 5$, additional non-perturbative states may play a role in the cosmological evolution. In fact, D5-branes of the type I theory (or NS5-branes in the heterotic context) can wrap the internal manifold in analogy with the D-strings we have considered.[6] It would be interesting to study their effects on the dynamics, which may lead eventually to a stabilization of the dilaton.

4 E1-instanton corrections

We have found that non-perturbative states contribute to the type I free energy density. In the literature, corrections to the low energy effective action are often considered from another point of view, namely instantons and their stringy generalizations. For instance, E1 contributions to holomorphic couplings have been analyzed in supersymmetric cases by heterotic/type I duality [50, 51]. In the present section, our aim is to reexamine the type I free energy from the point of view of E1-instantons and single out the configurations responsible for the stabilization of internal radii. In this non-supersymmetric case, we want to predict the E1 corrections in type I from dual heterotic worldsheet instantons. For simplicity, we restrict our analysis to the case $D = 9$, where instantons wrap the Euclidean time circle and the direction 9. This is to be contrasted with the zero temperature case where E1 corrections would only arise for $D \leq 8$. We note that by a double Wick rotation, the results in this section may be interpreted as the zero temperature vacuum energy contribution of E1-instantons wrapping an internal T^2, with spontaneous supersymmetry breaking boundary conditions along one of the toroidal directions. In this case, the temperature scale T is replaced with the supersymmetry breaking scale M.

Our starting point is the heterotic model of section 3. To help exhibit the worldsheet instanton structure of the one-loop amplitude $Z_{\rm h}$, we work in the Lagrangian formulation of the zero modes lattice associated to $S^1(R_{\rm h0}) \times S^1(R_{\rm h9})$ (see eqs. (A.10) and (A.11)). We consider $R_{\rm h9} \geq 1$ and parameterize the zero modes by the matrix $\mathcal{M} = \begin{pmatrix} n^0 & \tilde m_0 \\ n^9 & \tilde m_9 \end{pmatrix}$. The case $R_{\rm h9} \leq 1$ may be obtained by T-duality. We may decompose the lattice sum under orbits of the $\mathrm{SL}(2,\mathbb{Z})$ modular group as follows. For any set of modular covariant functions $f_{\mathcal{M}}(\tau,\bar\tau)$ such that $f_{\mathcal{M}}(M(\tau), M(\bar\tau)) = f_{\mathcal{M}M}(\tau,\bar\tau)$, for all $M \in \mathrm{SL}(2,\mathbb{Z})$, one has

$$\begin{aligned}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{\mathcal{M}} f_{\mathcal{M}}(\tau,\bar\tau) =& \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, f_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0\end{smallmatrix}\right)}(\tau,\bar\tau) \\ &+ \int_{\mathcal{S}_+} \frac{d^2\tau}{\tau_2^2} \sum_{\tilde m_0, \tilde m_9}{}' f_{\left(\begin{smallmatrix} 0 & \tilde m_0 \\ 0 & \tilde m_9\end{smallmatrix}\right)}(\tau,\bar\tau) + \int_{\mathbb{C}_+} \frac{d^2\tau}{\tau_2^2}\, 2 \sum_{\substack{\tilde m_0 \neq 0 \\ n^9 > \tilde m_9 \geq 0}} f_{\left(\begin{smallmatrix} 0 & \tilde m_0 \\ n^9 & \tilde m_9\end{smallmatrix}\right)}(\tau,\bar\tau).\end{aligned} \tag{4.1}$$

This is easily shown by applying eq. (A.6) twice: First to the sum over $(n^0, \tilde m_0)$ and then to the sum over $(n^9, \tilde m_9)$. The integral over the upper half plane $\mathbb{C}_+$ is obtained for $n^9 > 0$

[6] Note that these states may contribute even for $D = 5$. This is to be contrasted with 5-brane instantons at zero temperature, which require an internal space of six dimensions.

by writing $\tilde{m}_9 = kn^9 + l$ $(0 \le l < n^9 - 1)$ and changing $\tau \to \tau + k$. The integral over $\mathcal{F}$ corresponds to the zero orbit (i.e.$\mathcal{M} = 0$), while the integral over $\mathcal{S}_+$ corresponds to non-vanishing degenerate matrices (i.e.with $\det \mathcal{M} = 0$). The last integral over $\mathbb{C}_+$ is associated to non-degenerate matrices.

Applying (4.1) to the heterotic partition function $Z_{\rm h}$, the contribution of the zero orbit vanishes due to supersymmetry, so that[7]

$$\begin{aligned} Z_{\rm h} &= Z_{\rm h}^d + Z_{\rm h}^{nd} \\ Z_{\rm h}^d &= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^9}\int_{\mathcal{S}_+}\frac{d^2\tau}{2\tau_2^6}\,\frac{\Gamma_{(0,16)}}{\eta^8\bar{\eta}^{24}}\,R_9\sum_{\tilde{m}_0,\tilde{m}_9}{}' e^{-\frac{\pi R_{\rm h0}}{\tau_2}\tilde{m}_0^2}\,e^{-\frac{\pi R_{\rm h9}}{\tau_2}\tilde{m}_9^2}\Big[V_8 - (-1)^{\tilde{m}_0}S_8\Big] \\ Z_{\rm h}^{nd} &= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^9}\int_{\mathbb{C}_+}\frac{d^2\tau}{2\tau_2^6}\,\frac{\Gamma_{(0,16)}}{\eta^8\bar{\eta}^{24}}\,R_9\,2\sum_{\substack{\tilde{m}_0\neq 0\\ n^9>\tilde{m}_9\geq 0}} e^{-\frac{\pi R_{\rm h0}}{\tau_2}\tilde{m}_0^2}\,e^{-\frac{\pi R_{\rm h9}}{\tau_2}|n^9\tau+\tilde{m}_9|^2}\Big[V_8 - (-1)^{\tilde{m}_0}S_8\Big]. \end{aligned} \tag{4.2}$$

Performing the τ-integrations, the degenerate part $Z_{\rm h}^d$ can be brought into the form

$$Z_{\rm h}^d = \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{\hat{\beta}_{\rm h}^9}\left\{s_0 b_0\, c_9 + \sum_{A\geq 0,\, m_9}{}' s_A b_A\, G\!\left(2\pi R_{\rm h0}\left[4A + \left(\frac{m_9}{R_{\rm h9}}\right)^2\right]^{\frac{1}{2}}\right)\right\}, \tag{4.3}$$

while the non-degenerate contribution $Z_{\rm h}^{nd}$ can be written as,

$$Z_{\rm h}^{nd} = \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{\hat{\beta}_{\rm h}^9}\;2\sum_{\substack{A\geq 0,\,\bar{A}\geq -1\\ n^9>\tilde{m}_9\geq 0}} s_A b_{\bar{A}}\,\frac{e^{-2i\pi\frac{\tilde{m}_9}{n^9}(A-\bar{A})}}{n^9}\,G\!\left(2\pi R_{\rm h0}\left[4A + \left(\frac{A-\bar{A}}{n^9 R_{\rm h9}} - n^9 R_{\rm h9}\right)^2\right]^{\frac{1}{2}}\right). \tag{4.4}$$

Summing over $\tilde{m}_9$ in (4.4) enforces the level matching condition $A - \bar{A} = n^9 m_9$ for some integer m_9, whenever $n^9 \neq 0$. The "missing term" for $n^9 = 0$ is actually the contribution of the degenerate orbits $Z_{\rm h}^d$. In total, $Z_{\rm h}^d + Z_{\rm h}^{nd}$ yields with no surprise the expression (3.1), which can be analytically continued in the range $1 \le R_{\rm h9} \le \sqrt{2}$. However, to exhibit the instantonic structure, it is better to leave the sum over $\tilde{m}_9$.

In $Z_{\rm h}^d$, only pure Kaluza-Klein modes along the directions 9 and 0 contribute and the worldsheet embedding in the target torus is trivial (no instanton number). Therefore, these states do not play a role in stabilizing the internal circle. In order to extract the configurations in $Z_{\rm h}^{nd}$ responsible for fixing $R_{\rm h9}$ at the self-dual point, we know it is enough to focus on the dominant contributions in the low temperature expansion. The terms with $A \geq 1$ are exponentially suppressed, $\mathcal{O}(e^{-4\pi R_{\rm h0}})$, compared to the contribution with $A = 0$. The latter arises from BPS configurations and, at this level of approximation, $Z_{\rm h}^{nd}$ in eq. (4.2) involves a purely antiholomorphic function, $\mathcal{B}(\bar{\tau}) = \Gamma_{(0,16)}/\bar{\eta}^{24}$, dressed by an

[7] The use of eq. (4.1) is valid if the argument of the discrete sum to integrate is absolutely convergent. In the present case, since the right-moving block $\Gamma_{(0,16)}/\bar{\eta}^{24}$ and the left-moving O_8/η^8 character involve diverging powers of $e^{2\pi\tau_2}$ in the limit $\tau_2 \to +\infty$, eq. (4.1) can be trusted if $R_{\rm h0} > \sqrt{3}$ and $R_{\rm h9} > \sqrt{2}$. The first condition is not problematic as we are focussing on the dynamics at low temperature. Since we are interested in the stabilization of $R_{\rm h9}$ around 1, the second condition could be a problem. However, we see shortly that the final expression (4.4) can be analytically continued all the way to $R_{\rm h9} = 1$.

inverse power of τ_2 and the lattice of zero modes associated to the directions 0 and 9. This form is similar to the one encountered in the evaluation of holomorphic couplings, when supersymmetry is unbroken [50, 51].

We can now define instanton configurations, with associated Kähler and complex structure moduli Υ and $\mathcal{Y}$ as,

$$\textit{Instanton with}\quad n^9 > \tilde{m}_9 \geq 0\ ,\ \ \tilde{k}_0 \geq 0: \quad \begin{cases} \Upsilon = i\Upsilon_2 = i(2\tilde{k}_0+1)R_{h0}\cdot n^9 R_{h9} \\ \mathcal{Y} = \mathcal{Y}_1 + i\mathcal{Y}_2 = \dfrac{\tilde{m}_9}{n^9} + i\,\dfrac{(2\tilde{k}_0+1)R_{h0}}{n^9 R_{h9}} \end{cases}, \tag{4.5}$$

where $(2\tilde{k}_0{+}1)\,n^9$ is the instanton number, which counts the number of times the worldsheet wraps around the target torus. Using these notations and introducing coefficients $\alpha_n \in \mathbb{N}$ in the expansion of the Bessel function[8] in (2.2), $K_{\frac{9}{2}}(x) = \sqrt{\pi/(2x)}e^{-x}\sum_{n=0}^{4}\alpha_n/x^n$, we may write (4.4) as

$$\begin{aligned} Z_{\rm h}^{nd} = {} & \frac{\hat{V}_{\rm h}^{(10)}}{(2\pi)^{10}}\, 2 \sum_{\rm instantons} s_0\, \frac{e^{2i\pi\Upsilon}}{\Upsilon_2\, \mathcal{Y}_2^4} \sum_{n=0}^{4}\left[\frac{\alpha_n}{(2\pi\Upsilon_2)^n}\sum_{\bar{A}\geq -1} b_{\bar{A}}\left(1+\bar{A}\,\frac{\mathcal{Y}_2}{\Upsilon_2}\right)^{4-n} e^{2i\pi\mathcal{Y}\bar{A}}\right] \\ & + c.c. + \mathcal{O}(e^{-4\pi R_{h0}}), \end{aligned} \tag{4.6}$$

where $\hat{V}_{\rm h}^{(10)}$ is the ten-dimensional Euclidean volume. This result can be given a more elegant appearance by noting that $\mathcal{B}(\mathcal{Y})$ is a modular form of weight 4. Introducing the modular covariant derivative $D\mathcal{X} = (\partial_{\mathcal{Y}} + \frac{ir}{2\mathcal{Y}_2})\mathcal{X}$, where $\mathcal{X}(\mathcal{Y})$ is any modular form of weight r,[9] the brackets in (4.6) become $1/(\pi\Upsilon_2)^n\sum_{m=0}^{n}\gamma_{nm}(i\mathcal{Y}_2)^m D^m\mathcal{B}(\mathcal{Y})$, where γ_{nm} are rational numbers.

The above expression of $Z_{\rm h}^{nd}$ contains far too many explicit terms needed to study the stabilization of $R_{\rm h9}$. In (4.4), the dominant contribution for $A=0$ arises when $\bar{A}=-1$ and $n^9=1$, while the remaining terms are exponentially suppressed, $\mathcal{O}(e^{-2\pi R_{\rm h0}})$. Restricting to $\bar{A}=-1$ and the instanton configurations $n^9=1$, $\tilde{m}_9=0$, $\tilde{k}_0\geq 0$ in $Z_{\rm h}^{nd}$, we can add the degenerate contribution $Z_{\rm h}^{d} = (\hat{\beta}_{\rm h}\hat{V}_{\rm h}/\hat{\beta}_{\rm h}^{9})\, s_0 b_0\, c_9 + \mathcal{O}(e^{-2\pi R_{\rm h0}})$ to recover the first line of eq. (3.1) required for the derivation of the stabilization of $R_{\rm h9}$.

We now wish to interpret eq. (4.6) from the perspective of the type I superstring. Under the heterotic/type I dictionary (3.2), the complex and Kähler structures $\mathcal{Y}$ and Υ are mapped into $\mathcal{Y}_{\rm I}$ and $\Upsilon_{\rm I}/\lambda_{\rm I}$. Consequently, the exponential factor of Υ in (4.6) yields the exponential of the Nambu-Goto action for a D-string, and $Z_{\rm h}^{nd}$ translates into a sum of E1 instantons as in [50, 51],

$$\begin{aligned} Z_{\rm I}^{E1} = {} & \frac{\hat{V}_{\rm I}^{(10)}}{(2\pi)^{10}}\, 2 \sum_{\rm E1\,instantons} s_0\, \frac{e^{\frac{2i\pi}{\lambda_{\rm I}}\Upsilon_{\rm I}}}{\Upsilon_{\rm I2}\, \mathcal{Y}_{\rm I2}^4} \sum_{n=0}^{4}\left[\frac{\alpha_n}{(2\pi\Upsilon_{\rm I2})^n}\sum_{\bar{A}\geq -1} b_{\bar{A}}\left(\frac{1}{\lambda_{\rm I}}+\bar{A}\,\frac{\mathcal{Y}_{\rm I2}}{\Upsilon_{\rm I2}}\right)^{4-n} e^{2i\pi\mathcal{Y}_{\rm I}\bar{A}}\right] \\ & + c.c. + \mathcal{O}(e^{-4\pi\frac{R_{\rm I0}}{\sqrt{\lambda_{\rm I}}}}). \end{aligned} \tag{4.7}$$

[8] In any odd dimension, the Bessel function admits a power series with a finite number of terms.

[9] This means that $\mathcal{X}(\mathcal{Y}+1) = \mathcal{X}(\mathcal{Y})$ and $\mathcal{X}(-1/\mathcal{Y}) = \mathcal{X}(\mathcal{Y})/\mathcal{Y}^r$. Moreover, $D\mathcal{X}$ is a modular form of weight $r-2$.

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Actually, the configurations of the D-string worldsheets wrapped on $S^1(R_{\rm I0})\times S^1(R_{\rm I9})$ are highly dissymmetric at late times in the sense that $R_{\rm I0}(t)\to+\infty$ and $R_{\rm I9}(t)\sim\sqrt{\lambda_{\rm I}(t)}\to\sqrt{\lambda_{\rm I0}}$. However, this does not mean it is artificial to consider such E1-instantons. Instead, they open the possibility to derive from a pure type I point of view the free energy responsible for the stabilization of the internal moduli (or the effective potential at zero temperature when at least two internal directions are compactified and supersymmetry is spontaneously broken). Thus, it would be interesting to derive D-brane instanton corrections from first principles, in the case where supersymmetry is spontaneously broken. The full instantonic structure of (4.4) should also be interpreted from a type I point of view, even when all contributions with $A\geq 0$ and $\bar A\geq -1$ are kept explicitly.

5 Heterotic and dual type I moduli stabilization

We would like to extend the analysis used in section 3 to include the remaining moduli in addition to the internal radii. We consider the heterotic string compactified on T^{10-D} at a generic point in moduli space and show that when finite temperature is switched on, the free energy density can stabilize all internal moduli. Our study is based on the effective action at finite temperature and weak coupling for the massless degrees of freedom, while all massive states are integrated out. Introducing simplified notations, we are interested in non-trivial backgrounds for the Einstein frame metric g, the dilaton ϕ in D dimensions and all real-valued internal moduli Φ^M, which we denote collectively as $\vec\Phi$. Concretely, $\vec\Phi$ contains the components of the metric $\hat g_{ij}$ and antisymmetric tensor B_{ij}, together with the Wilson lines Y_i^I $(i,j=D,\dots,9;\ I=10,11,\dots,25)$. It is then straightforward to deduce the dynamics and final expectation values of the type I counterparts of these scalars by using the duality map

$$\hat g_{ij}=\frac{\hat g_{{\rm I}ij}}{\lambda_{\rm I}}\;,\qquad\qquad B_{ij}=C_{ij}\;,\qquad\qquad Y_i^I=Y_{{\rm I}i}^I\;,\tag{5.1}$$

where C_{ij} is the RR 2-form. Detailed examples of this analysis will be given in section 6 for $D=8$.

The heterotic low energy effective action

$$S=\int d^Dx\sqrt{-g}\left[\frac{R}{2}-\frac{2}{D-2}\,\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}F_{MN}\partial_\mu\Phi^M\partial^\mu\Phi^N-\mathcal{F}\right]\tag{5.2}$$

involves the tree level moduli space metric $F_{MN}=F_{MN}(\vec\Phi)$ and the one-loop free energy density $\mathcal{F}=\mathcal{F}(T,\phi,\vec\Phi)$. Since the backreaction of $\mathcal{F}$ on the classical background is already a one-loop effect, there is no need to take into account the quantum corrections to the kinetic terms. For homogeneous and isotropic evolutions, variation of S with respect to the time-dependent metric (3.3), dilaton and moduli Φ^M yields, in cosmological time defined

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by $dt \equiv \beta(x^0)dx^0$,

$$\frac{(D-1)(D-2)}{2}\, H^2 = \frac{2}{D-2}\,\dot\phi^2 + \frac{1}{2}\, F_{MN}\dot\Phi^M\dot\Phi^N + \rho, \tag{5.3}$$

$$\frac{(D-1)(D-2)}{2}\, H^2 + (D-2)\dot H + \frac{2}{D-2}\,\dot\phi^2 + \frac{1}{2}\, F_{MN}\dot\Phi^M\dot\Phi^N + P = 0, \tag{5.4}$$

$$\ddot\phi + (D-1)H\dot\phi + \frac{D-2}{4}\,\mathcal{F}_\phi = 0, \tag{5.5}$$

$$\ddot\Phi^M + (D-1)H\dot\Phi^M + F^{MN}\Big(F_{NPQ} - \frac{1}{2}F_{PQN}\Big)\dot\Phi^P\dot\Phi^Q + F^{MN}\mathcal{F}_N = 0. \tag{5.6}$$

In these equations, $H = \dot a/a$ and the thermal pressure and energy density are found to be

$$P = -\mathcal{F}\,, \qquad\qquad \rho = T\,\frac{\partial P}{\partial T} - P. \tag{5.7}$$

Additional indices ϕ and N denote partial derivatives with respect to ϕ and Φ^N, while $F^{MN} \equiv (F^{-1})^{MN}$. It is convenient to replace eq. (5.4) by the constant entropy constraint. The latter is found by integrating the energy-momentum tensor conservation law derived from the above differential system (see [31]),

$$\dot\rho + (D-1)H(\rho+P) = \dot\phi\,\mathcal{F}_\phi + \dot\Phi^M\mathcal{F}_M \quad\Longrightarrow\quad a^{D-1}\,\frac{\rho+P}{T} = \text{constant entropy}. \tag{5.8}$$

In order to find particular evolutions characterized by static moduli, $\big(\phi(t),\vec\Phi(t)\big) \equiv (\phi_0,\vec\Phi_0)$, we need to specify $\mathcal{F}$. For any supersymmetric spectrum, the one-loop free energy density is

$$\mathcal{F} = -e^{\frac{2D}{D-2}\phi}\int_0^{+\infty}\frac{dl}{2l}\,\frac{1}{(2\pi l)^{\frac{D}{2}}}\sum_s e^{-\frac{\hat M_s^2 l}{2}}\sum_{\tilde m_0} e^{\frac{-\hat\beta^2\tilde m_0^2}{2l}}\left(1-(-)^{\tilde m_0}\right) = -T^D\sum_s G\Big(\frac{e^{\frac{2}{D-2}\phi}\hat M_s}{T}\Big), \tag{5.9}$$

where $\hat M_s$ is the mass of each boson/fermion pair s, and the dilaton dressing in front of the integral is introduced to switch from string to Einstein frame. This general expression applied to our case of interest, namely the heterotic string on T^{10-D}, is explicitly derived from a one-loop vacuum-to-vacuum amplitude in the appendix. In the notations introduced there, $s_0 r_0 = 2^8 \times 24$ boson/fermion pairs of states are massless everywhere in moduli space,[10] while the other modes have moduli-dependent masses, $\hat M_s(\vec\Phi)$. Since light states have the tendency to lower $\mathcal{F}$, effective potential wells can be found at any point $\vec\Phi_0$ where $n_0 > 0$ pairs of modes generically massive are accidentally massless, $\hat M_u(\vec\Phi_0) = 0$, $u = 1,\dots,n_0$. The fact that we have at zero temperature 16 real conserved supercharges implies that such points are associated to enhancements of the gauge symmetry. Defining $\hat M_{\rm min}$ to be the lightest non-vanishing mass at $\vec\Phi_0$, the free energy density can be written in a neighborhood of $\vec\Phi_0$ as,

$$\mathcal{F} = -T^D\left\{s_0 r_0 + \sum_{u=1}^{n_0} G\Big(\frac{e^{\frac{2}{D-2}\phi}\hat M_u(\vec\Phi)}{T}\Big) + \mathcal{O}\big(e^{-\frac{\hat M_{\rm min}}{T}}\big)\right\}. \tag{5.10}$$

[10]They are associated to the supergravity and super-vector multiplets of the SO(32) Cartan generators.

At low enough temperature, the exponentially suppressed terms can be neglected and we may derive identities for the thermal source terms at $\vec{\Phi}_0$, including the equation of state,

$$\mathcal{F}|_{\vec{\Phi}_0} = -T^D(s_0 r_0 + n_0)\, c_D\,, \quad \mathcal{F}_\phi|_{\vec{\Phi}_0} = 0\,, \quad \mathcal{F}_M|_{\vec{\Phi}_0} = 0\,, \quad \rho|_{\vec{\Phi}_0} = (D-1)P|_{\vec{\Phi}_0} \propto T^D. \tag{5.11}$$

It is then straightforward to check that the evolutions

$$a_0(t) \propto \frac{1}{T_0(t)} \propto t^{2/D}\,, \qquad \phi(t) \equiv \phi_0\,, \qquad \vec{\Phi}(t) \equiv \vec{\Phi}_0\,, \tag{5.12}$$

corresponding to radiation eras with static moduli are particular solutions of the equations of motion.

The above trajectories are actually attractors of the dynamics in some circumstances. To study this, we analyze their stability under small time-dependent deviations,

$$a = a_0(1+\epsilon_a)\,, \qquad T = T_0(1+\epsilon_T)\,, \qquad \phi = \phi_0 + \epsilon_\phi\,, \qquad \Phi^M = \Phi_0^M + \epsilon^M. \tag{5.13}$$

We first perturb the internal moduli equation (5.6). Denoting $H_0 = \dot{a}_0/a_0$, one obtains at lowest order,

$$\ddot{\epsilon}^M + (D-1)H_0\,\dot{\epsilon}^M + \Lambda^M{}_N\,\epsilon^N = 0 \qquad \text{where} \qquad \Lambda^M{}_N \equiv F^{ML}|_{\vec{\Phi}_0}\mathcal{F}_{LN}|_{(T_0,\phi_0,\vec{\Phi}_0)}. \tag{5.14}$$

$\Lambda^M{}_N$ is an effective "time-dependant squared mass matrix" evaluated for the background (5.12). Since

$$\mathcal{F}_{MN}|_{\vec{\Phi}_0} = T^{D-2}\, e^{\frac{4\phi}{D-2}}\, \frac{c_{D-2}}{4\pi} \sum_{u=1}^{n_0} \left.\frac{\partial^2 \hat{M}_u^2}{\partial\Phi^M \partial\Phi^N}\right|_{\vec{\Phi}_0} \tag{5.15}$$

is semi-definite positive, $\Lambda^M{}_N$ is diagonalizable with non-negative eigenvalues,[11] which we define as $\frac{4\lambda_M^2}{D^2 t^{2(D-2)/D}}$. In the case when some λ_M's vanish, one needs to take into account quadratic terms in eq. (5.14) (see the discussion of the dilaton equation below). In particular, this is required when moduli sit on the plateau of their thermal effective potential (see figure 1). For simplicity, we proceed by analyzing the most interesting case, where all internal moduli are "massive", which means $\lambda^M > 0$. Switching to a diagonal basis of perturbations $\tilde{\epsilon}^M$, one obtains from (5.14)

$$\tilde{\epsilon}^M = \frac{t^{1/D}}{\sqrt{t}} \left[C_+^M\, J_{\frac{D-2}{4}}(\lambda_M\, t^{2/D}) + C_-^M\, J_{-\frac{D-2}{4}}(\lambda_M\, t^{2/D}) \right], \tag{5.16}$$

where $C_\pm^M$ are integration constants and $J_{\pm\frac{D-2}{4}}$ are Bessel functions of the first kind.[12] This describes damped oscillations with amplitude of order $1/\sqrt{t}$, where t is supposed to be large enough so that $|\tilde{\epsilon}^M| \ll 1$ is satisfied.

[11]This follows from the fact that the matrices $F^{-1/2}$ and $\mathcal{F}$ are (semi-)definite positive, so that $F^{-1/2}\mathcal{F}F^{-1/2} = F^{1/2}\Lambda F^{-1/2}$ is semi-definite positive. Note that in models where the spontaneous breaking of supersymmetry is generic i.e.not only due to thermal effects, each term in the sum over the boson-fermion pair u in eq. (5.15) is dressed with a $+$ (or $-$) sign when the boson (fermion) is lighter than the fermion (boson). In such cases, $\mathcal{F}$ is not semi-definite positive and the extrema of $\mathcal{F}$ can be minima, maxima or saddle points.

[12]For $D = 6$, J_{-1} should be replaced by the Bessel function of the second kind, Y_{-1}.

Next, we derive from (5.5) the equation for the dilaton perturbation at leading order,

$$\left(a_0^{D-1}\dot\epsilon_\phi\right)^{\dot{}}+a_0^{D-1}\,\frac{1}{2}\,\mathcal{F}_{\phi MN}|_{(T_0,\phi_0,\vec\Phi_0)}\epsilon^M\epsilon^N=0\quad\text{where}\quad \mathcal{F}_{\phi MN}|_{\vec\Phi_0}\equiv\frac{4}{D-2}\,\mathcal{F}_{MN}|_{\vec\Phi_0}. \tag{5.17}$$

Since the constants $C_\pm^M$ are a priori of order one, we take into account the quadratic source in "massive" epsilons. Thus, $\dot\epsilon_\phi$ can be written as the sum of the general solution to its homogeneous equation, plus a particular solution to eq. (5.17). The former is of order $1/a_0^{D-1}$ and turns out to be dominated at late times by the latter. Actually, using (5.16), the quadratic source term involves products of Bessel functions with arguments $\lambda_P\,t^{2/D}$ and $\lambda_Q\,t^{2/D}$. Integrating it once, the dominant contribution to $a_0^{D-1}\dot\epsilon_\phi$ is found to arise for "constructive interferences", i.e.when $\lambda_P=\lambda_Q$. This yields the asymptotic behavior,

$$\dot\epsilon_\phi\sim-\frac{C_\phi}{a_0^{D-2}}\qquad\Longrightarrow\qquad \epsilon_\phi\propto\frac{1}{t^{1-4/D}}\;\text{ for }\;D\geq 5\quad\text{and}\quad \epsilon_\phi\propto\ln t\;\text{ for }D=4, \tag{5.18}$$

where C_ϕ is a fully determined coefficient quadratic in $C_\pm^M$'s and positive. For $D\geq 5$, the consistency condition $|\epsilon_\phi|\ll 1$ is automatically fulfilled at late times. On the contrary, the case $D=4$ yields formally to a logarithmically decreasing ϵ_ϕ and one may worry that the our expansions breaks down. Therefore, we have solved numerically the full non-linear differential system (5.3)–(5.6) in this case and found that the perturbative analysis gives the correct late time behavior, which we summarize at the end of this section.

To analyze the evolution of the scale factor and temperature fluctuations, we expand the energy density and pressure around the background (5.12) and find from Friedmann's equation (5.3) and (5.8),

$$(D-1)(D-2)\,H_0\,\dot\epsilon_a=\frac{1}{2}\,F_{MN}|_{\vec\Phi_0}\dot\epsilon^M\dot\epsilon^N+D\rho|_{(T_0,\phi_0,\vec\Phi_0)}\epsilon_T-\frac{D-3}{2}\,\mathcal{F}_{MN}|_{(T_0,\phi_0,\vec\Phi_0)}\epsilon^M\epsilon^N, \tag{5.19}$$

$$D(\epsilon_a+\epsilon_T)\,\rho|_{(T_0,\phi_0,\vec\Phi_0)}=\frac{D-2}{2}\,\mathcal{F}_{MN}|_{(T_0,\phi_0,\vec\Phi_0)}\epsilon^M\epsilon^N. \tag{5.20}$$

It is then straightforward to solve for ϵ_a, whose asymptotic behavior is again dictated by the source terms in "constructive interferences" arising from the products $\dot\epsilon^M\dot\epsilon^N$ and $\epsilon^M\epsilon^N$ in (5.19) and (5.20). The late time scaling property of ϵ_a is found to be

$$\epsilon_a\propto\frac{a_0^2}{t}\propto\frac{1}{t^{1-4/D}}, \tag{5.21}$$

which can be used in eq. (5.20) to find

$$\epsilon_T\propto\frac{a_0^2}{t}\,(1+\text{oscillations with constant amplitude}). \tag{5.22}$$

In (5.21) and (5.22), the coefficients of proportionality are again fully expressed in terms of the $C_\pm^M$'s.

We signal that for $D\geq 5$, all terms we have neglected in the perturbed equations of motion are a posteriori found to be dominated by the sources we took into account.

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This guarantees the validity of the asymptotic behaviors we have found for the deviations defined in (5.13). These results have been confirmed by direct numerical analysis of the unperturbed system of differential equations in some examples. Since all fluctuations converge to zero, the late time cosmology is radiation dominated. In particular, the dilaton motion and the damped oscillations of $\tilde{\epsilon}^M$ store a negligible amount of energy as compared to the thermal radiation energy. The internal moduli are dynamically stabilized and their effective time-dependent masses (measured in Einstein frame) are $M_{\tilde{\Phi}^M} \propto T_0^{\frac{D-2}{2}} e^{\frac{2\phi_0}{D-2}}$.

For $D = 4$, the numerical simulations show that the internal moduli converge to $\vec{\Phi}_0$, while the dilaton decreases logarithmically with time. Individually, the energy stored in the dilaton motion, the total energy (kinetic plus potential) of the damped oscillations of $\vec{\Phi}$, and the thermal radiation energy decay at the same rate. Their late time behavior satisfies

$$H^2 \propto \dot{\phi}^2 \propto \left(\frac{1}{2}\, F_{MN}\dot{\Phi}^M\dot{\Phi}^N + \rho\right) \propto \frac{1}{a^4}, \tag{5.23}$$

so that the metric evolution is identical to that of a radiation dominated universe, $a \propto \sqrt{t}$.

The above logarithmic behavior of the heterotic dilaton is transferred by heterotic/type I duality to the type I dilaton for $D = 4$. Moreover, in any dimension, stabilization of the internal moduli on the heterotic side implies stabilization of internal moduli on the type I side, except for the special case of $D = 6$, where S-duality exchanges the six-dimensional heterotic coupling with the type I internal volume modulus.

6 Example: dual heterotic/type I strings on T^2

Our aim is to illustrate the analysis of the previous section with examples for $D = 8$. We want to find local attractor solutions of the form (5.12) associated to enhanced symmetry points $\vec{\Phi}_0$ of the internal moduli space of the heterotic string on T^2. We shall see that the one-loop free energy density has enough structure to stabilize $\mathcal{T} = B_{89} + i\sqrt{\hat{g}_{88}\hat{g}_{99} - \hat{g}_{89}^2}$, $\mathcal{U} = \big(\hat{g}_{89} + i\sqrt{\hat{g}_{88}\hat{g}_{99} - \hat{g}_{89}^2}\big)/\hat{g}_{88}$ and the Wilson lines Y_i^I $(i, j = 8, 9;\ I = 10, 11, \dots, 25)$. This translates in the type I side into expectation values of the closed and open string internal moduli via the duality map $\mathcal{T} = \mathcal{T}_{\rm I}$, $\mathcal{U} = \mathcal{U}_{\rm I}$, $Y_i^I = Y_{{\rm I}i}^I$, where

$$\begin{aligned}
\mathcal{T}_{\rm I} &= C_{89} + i\frac{\sqrt{\hat{g}_{\rm I88}\hat{g}_{\rm I99} - \hat{g}_{\rm I89}^2}}{\lambda_{\rm I}} = C_{89} + ie^{-\phi_{\rm I}}\frac{\left(\hat{g}_{\rm I88}\hat{g}_{\rm I99} - \hat{g}_{\rm I89}^2\right)^{1/4}}{2\pi}\,, \\
\mathcal{U}_{\rm I} &= \frac{\hat{g}_{\rm I89} + i\sqrt{\hat{g}_{\rm I88}\hat{g}_{\rm I99} - \hat{g}_{\rm I89}^2}}{\hat{g}_{\rm I88}}.
\end{aligned} \tag{6.1}$$

The only remaining flat direction of the thermal effective potential corresponds to the heterotic and type I dilatons in eight dimensions, which are related as: $\phi_{\rm I} = -\frac{1}{2}\phi - \frac{3}{4}\ln\big((2\pi)^2\sqrt{\hat{g}_{88}\hat{g}_{99} - \hat{g}_{89}^2}\big)$.

The heterotic effective action in the Einstein frame is (see for instance appendices D and E in [56])

$$S=\int d^8x\sqrt{-g}\left\{\left[\frac{R}{2}-\frac{(\partial\phi)^2}{3}-\frac{1}{4}\left(\frac{|\partial\mathcal{U}|^2}{\mathcal{U}_2^2}+\frac{|\partial\mathcal{T}+Y^I_{[8}\partial Y^I_{9]}|^2}{\mathcal{T}_2^2}+\frac{|\mathcal{U}\partial Y^I_8-\partial Y^I_9|^2}{\mathcal{T}_2\mathcal{U}_2}\right)\right]-\mathcal{F}\right\}. \tag{6.2}$$

Indeed, if we arrange the thirty-four entries of the moduli vector as $\vec{\Phi}\equiv(\mathcal{T}_1,\mathcal{T}_2,\mathcal{U}_1,\mathcal{U}_2,Y^I_8,Y^{I'}_9)$, where indices 1 and 2 refer to real and imaginary parts, the metric components of the general expression (5.2) are

$$(F_{MN})=\begin{pmatrix}\frac{1}{2\mathcal{T}_2^2} & 0 & 0 & 0 & -\frac{Y^J_9}{4\mathcal{T}_2^2} & \frac{Y^{J'}_8}{4\mathcal{T}_2^2}\\ & \frac{1}{2\mathcal{T}_2^2} & 0 & 0 & 0 & 0\\ & & \frac{1}{2\mathcal{U}_2^2} & 0 & 0 & 0\\ & & & \frac{1}{2\mathcal{U}_2^2} & 0 & 0\\ & \text{sym.} & & & \frac{|\mathcal{U}|^2}{2\mathcal{T}_2\mathcal{U}_2}\delta^{IJ}+\frac{Y^I_9Y^J_9}{8\mathcal{T}_2^2} & -\frac{\mathcal{U}_1}{2\mathcal{T}_2\mathcal{U}_2}\delta^{IJ'}-\frac{Y^I_9Y^{J'}_8}{8\mathcal{T}_2^2}\\ & & & & & \frac{1}{2\mathcal{T}_2\mathcal{U}_2}\delta^{I'J'}+\frac{Y^{I'}_8Y^{J'}_8}{8\mathcal{T}_2^2}\end{pmatrix}. \tag{6.3}$$

The free energy density $\mathcal{F}$ is determined by the mass spectrum (see eq. (5.9)), which is specified by the left (right)-moving oscillator number A ($\bar{A}$), the internal momenta and winding numbers m_i, n^i ($i=8,9$), and the root vector Q^I of the right-moving internal lattice $\Gamma_{Spin(32)/\mathbb{Z}_2}$. As reviewed in the appendix, the mass formula $\hat{M}^2_s=2(A+\bar{A})+\frac{1}{2}\left(\vec{p}^{\,2}_L+\vec{p}^{\,2}_R\right)$ involves the left and right-moving momenta along the compact directions,

$$p^I_{L,R}=\left(m_i-Q^JY^J_i-n^jB_{ij}-\frac{1}{2}n^jY^J_iY^J_j\right)e^{*iI}\mp n^ie^I_i \text{ for } i,j,I=8,\dots,9;\ J=10,\dots,25,$$

$$p^I_R=\sqrt{2}\left(Q^I+n^iY^I_i\right) \text{ for } I=10,\dots,25;\ \vec{Q}\in\Gamma_{Spin(32)/\mathbb{Z}}, \tag{6.4}$$

where $\hat{g}_{ij}=e^I_ie^I_j$ and $e^{*iI}e^I_j=\delta^i_j$. More explicitly, one obtains

$$\hat{M}^2_{A,\vec{m},\vec{n},\vec{Q}}(\mathcal{T},\mathcal{U},Y)=\frac{1}{\mathcal{T}_2\mathcal{U}_2}\left|-m_8\mathcal{U}+m_9+\tilde{\mathcal{T}}n^8+\left(\tilde{\mathcal{T}}\mathcal{U}-\frac{1}{2}\mathcal{W}^I\mathcal{W}^I\right)n^9+\mathcal{W}^IQ^I\right|^2+4A, \tag{6.5}$$

where we have defined

$$\mathcal{W}^I:=\mathcal{U}Y^I_8-Y^I_9\ ,\qquad \tilde{\mathcal{T}}:=\mathcal{T}+\frac{1}{2}Y^I_8\mathcal{W}^I \tag{6.6}$$

and used the level matching condition, $A-\bar{A}=m_in^i+\frac{1}{2}Q^IQ^I$. At generic points in moduli space, the gauge symmetry is $\mathrm{U}(1)^2_L\times\mathrm{U}(1)^2_R\times\mathrm{U}(1)^{16}_R$, where $\mathrm{U}(1)^2_L\times\mathrm{U}(1)^2_R$ arises from T^2 compactification, and $\mathrm{U}(1)^{16}_R$ is the Cartan subgroup of $\mathrm{SO}(32)_R$. We now examine special points in moduli space where n_0 pairs of bosonic and fermionic superpartners generically massive are accidentally massless. Since at zero temperature the model is maximally supersymmetric, such points are associated to enhanced gauge symmetries. In fact, the additional massless modes arise at oscillator levels $A=0$, $\bar{A}=-1$, so that n_0 is proportional to $s_0 r_{-1} = 2^3$ (see the appendix) and the enhancements of the gauge theory arise from the right-moving sector only. In the following two examples, we will simplify the notations by omitting the subscript "R" in the right-moving gauge group factors.

Local attractor 1: $\mathbf{U(1)_L^2 \times SU(3) \times SO(32)}$. We start with the most obvious attractor where all Wilson lines vanish, $Y_i^I = 0$, leaving the SO(32) group unbroken. The torus moduli take the values $\mathcal{T} = \mathcal{U} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, implying an additional SU(3) gauge factor. The n_0 states responsible for the enhancement of $U(1)^2 \times U(1)^{16} \to SU(3) \times SO(32)$ are divided into two groups:

- 6×2^3 boson/fermion pairs imply $U(1)^2 \to SU(3)$. Their quantum numbers are $(\vec{m}, \vec{n}) = \pm(1,1;0,1)$, $\pm(0,1;-1,1)$ or $\pm(1,0;1,0)$, and $\vec{Q} = 0$. In this case, $p_L^{8,9} = 0 = p_R^{I\geq 10}$ and (p_R^8, p_R^9) realize the root vectors of SU(3), which represent a hexagon. The corresponding 6 mass formulas are,

$$\hat{M}^2_{0,\vec{m},\vec{n},\vec{0}} = \begin{cases} \frac{1}{\mathcal{T}_2\mathcal{U}_2}|1 - \mathcal{U} + \tilde{\mathcal{T}}\mathcal{U} - \frac{1}{2}\mathcal{W}^I\mathcal{W}^I|^2 & \text{for } (\vec{m},\vec{n}) = \pm(1,1,0,1), \\ \frac{1}{\mathcal{T}_2\mathcal{U}_2}|1 - \tilde{\mathcal{T}} + \tilde{\mathcal{T}}\mathcal{U} - \frac{1}{2}\mathcal{W}^I\mathcal{W}^I|^2 & \text{for } (\vec{m},\vec{n}) = \pm(0,1,-1,1), \\ \frac{1}{\mathcal{T}_2\mathcal{U}_2}|\tilde{\mathcal{T}} - \mathcal{U}|^2 & \text{for } (\vec{m},\vec{n}) = \pm(1,0,1,0). \end{cases} \tag{6.7}$$

- 480×2^3 boson/fermion pairs to recover $U(1)^{16} \to SO(32)$. They have $(\vec{m}, \vec{n}) = (\vec{0}, \vec{0})$, $\vec{Q} = \pm(1, \pm 1, 0, \dots, 0)$, $\pm(1, 0, \pm 1, \dots, 0)$ or any other permutation. In this case, $p_L^{8,9} = 0 = p_R^{8,9}$, while $(p_R^{I\geq 10})$ realize the root vectors of SO(32). The corresponding 480 mass formulas are

$$\hat{M}^2_{0,\vec{0},\vec{0},\vec{Q}} = \frac{1}{\mathcal{T}_2\mathcal{U}_2}\left|\pm(\mathcal{W}^I \pm \mathcal{W}^J)\right|^2 , \qquad I, J = 10, \dots, 25\ ,\ I \neq J. \tag{6.8}$$

To compute the squared mass matrix defined in eq. (5.14), we first evaluate the second derivatives (5.15) of the free energy at $\vec{\Phi}_0$. The non vanishing components are proportional to

$$\begin{aligned} &\sum_{u=1}^{n_0} \left.\frac{\partial^2 \hat{M}_u^2}{\partial \mathcal{T}_\alpha \partial \mathcal{T}_\alpha}\right|_{\vec{\Phi}_0} = \sum_{u=1}^{n_0} \left.\frac{\partial^2 \hat{M}_u^2}{\partial \mathcal{U}_\alpha \partial \mathcal{U}_\alpha}\right|_{\vec{\Phi}_0} = 16 \times 2^3, \qquad \alpha = 1,2 \text{ (no sum over } \alpha) \\ &\sum_{u=1}^{n_0} \left.\frac{\partial^2 \hat{M}_u^2}{\partial Y_i^I \partial Y_i^I}\right|_{\vec{\Phi}_0} = -2 \sum_{u=1}^{n_0} \left.\frac{\partial^2 \hat{M}_u^2}{\partial Y_8^I \partial Y_9^I}\right|_{\vec{\Phi}_0} = 160 \times 2^3,\ \ i = 8, 9; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad I = 10, \dots, 25 \text{ (no sum over } i, I). \end{aligned} \tag{6.9}$$

The nonzero entries of the metric (6.3) at $\vec{\Phi}_0$ are also found to be

$$\begin{aligned} F_{\mathcal{T}_\alpha \mathcal{T}_\alpha} &= F_{\mathcal{U}_\alpha \mathcal{U}_\alpha} = \frac{2}{3}, \qquad \alpha = 1,2 \text{ (no sum over } \alpha) \\ F_{Y_i^I Y_i^I} &= \frac{2}{3}\ , \quad F_{Y_8^I Y_9^I} = -\frac{1}{3}, \quad i = 8, 9;\ I = 10, \dots, 25 \text{ (no sum over } i, I). \end{aligned} \tag{6.10}$$

The resulting matrix of squared masses is diagonal, with strictly positive eigenvalues. Therefore, all flat directions of the internal moduli space are lifted. Once the dynamics is attracted to the trajectory (5.12), the "time-dependent moduli squared masses" are

$$M^2_{\Phi_1} = \frac{c_6}{4\pi}\, 2^3 \times 24\, e^{\frac{2\phi_0}{3}}\, T_0^6 \qquad \text{or} \qquad M^2_{\Phi_2} = \frac{c_6}{4\pi}\, 2^3 \times 240\, e^{\frac{2\phi_0}{3}}\, T_0^6\,. \tag{6.11}$$

The first one corresponds to $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{U}_1$, $\mathcal{U}_2$, while the second is associated to the Wilson lines Y_8^I and Y_9^I. The additional factor of ten for the latter can be understood from the fact that they are coupled to ten times as many additional states as compared to the torus moduli.

Local attractor 2: $\mathbf{U(1)}_L^2 \times \mathbf{SU(2)} \times \mathbf{SO(34)}$. The point $\vec{\Phi}_0$ we now consider corresponds to the values $\mathcal{T} = \mathcal{U} = i/\sqrt{2}$, $Y_8^{I\geq 10} = 0$ and $Y_9^{10} = -Y_9^{11} = -Y_9^{12} = \cdots = -Y_9^{25} = -1/2$. This moduli configuration is much less trivial than the previous one, since it is going to give rise to the gauge group $SU(2)_8 \times SO(34)_{9,\dots,25}$, where the subscripts denote which directions $i = 8, 9$ and $I = 10, \dots, 25$ are associated with the gauge factors. There are $n_0 = 546 \times 2^3$ extra massless boson/fermion pairs of states, which can be divided into 2×2^3 for the $SU(2)_8$ and 544×2^3 for the $SO(34)_{9,\dots,25}$ enhancements. Note that the $SO(34)_{9,\dots,25}$ factor arises from an enhancement of the $U(1)_9$ symmetry of the T^2 torus, with the $SO(32)$ symmetry of the internal lattice. The detailed quantum numbers of the extra states are as follows:

- 2×2^3 boson/fermion pairs give $U(1)_8 \to SU(2)_8$. They have $(\vec{m}, \vec{n}) = \pm(1, 0; 1, 0)$ and $\vec{Q} = 0$. In this case, $p_L^{I\geq 8} = 0 = p_R^{J\geq 9}$, while $p_R^8 = \pm\sqrt{2}$ realize the root vectors of $SU(2)_8$.

  For $SO(34)_{9,\dots,25}$, the 544×2^3 pairs of bosons and fermions giving $U(1)^{17}_{9,\dots,25} \to SO(34)_{9,\dots,25}$ are subdivided into:

- 420×2^3 pairs transform in the adjoint representation of $SO(30)$ and are giving rise to $U(1)^{15}_{11,\dots,25} \to SO(30)_{11,\dots,25}$. 210×2^3 have $(\vec{m}, \vec{n}, \vec{Q}) = \pm(0, 1; 0, 0; 0, 1, 1, 0, \dots, 0)$ or any permutation of the last 15 entries. The other 210×2^3 have $(\vec{m}, \vec{n}, \vec{Q}) = (0, 0; 0, 0; 0, 1, -1, 0, \dots, 0)$ or any permutation of the last 15 entries.

- 60×2^3 pairs transform as $(2, 30)$ under $SO(2)_{10} \times SO(30)_{11,\dots,25}$, giving the enhanced group $SO(32)_{10,\dots,25}$. 30×2^3 of them have $(\vec{m}, \vec{n}, \vec{Q}) = \pm(0, 1; 0, 0; -1, 1, 0, \dots, 0)$ or any permutation of the last 15 entries. The other 30×2^3 have $(\vec{m}, \vec{n}, \vec{Q}) = \pm(0, 0; 0, 0; 1, 1, 0, \dots, 0)$ or any permutation of the last 15 entries.

- 64×2^3 pairs transform as $(2, 32)$ under $SO(2)_9 \times SO(32)_{10,\dots25}$, giving the enhanced gauge group $SO(34)_{9,\dots,25}$. 32×2^3 of them have $(\vec{m}, \vec{n}, \vec{Q}) = \pm(0, 1; 0, -1; \frac{1}{2}, \dots, \frac{1}{2})$ and $\pm(0, 1; 0, -1; -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ or any permutation of the last 15 entries. The other 32×2^3 have $(\vec{m}, \vec{n}, \vec{Q}) = \pm(0, 2; 0, -1; -\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ and $\pm(0, 2; 0, -1; -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ or any permutation of the last 15 entries.

Proceeding as before, the squared mass matrix in (5.14) can be evaluated. Its diagonalization reveals two groups of eigenvalues,

$$M^2_{\tilde{\Phi}_1} = \frac{c_6}{4\pi}\, 2^3 \times 16\, e^{\frac{2\phi_0}{3}}\, T_0^6\,, \qquad M^2_{\tilde{\Phi}_2} = \frac{c_6}{4\pi}\, 2^3 \times 256\, e^{\frac{2\phi_0}{3}}\, T_0^6\,. \tag{6.12}$$

The first one is associated to $\mathcal{T}_1 - \mathcal{U}_1 - \frac{1}{4}(Y_8^{10} - Y_8^{11} - \cdots - Y_8^{25})$ and $\mathcal{T}_2 - \mathcal{U}_2$, while the second corresponds to $\mathcal{T}_1 + \mathcal{U}_1$, $\mathcal{T}_2 + \mathcal{U}_2$ and all 32 Wilson lines. Thus, we find a second point in moduli space where all internal moduli are stabilized by the thermal effective potential.[13]

7 Conclusions and perspectives

In this paper, we considered toroidally compactified heterotic and type I superstrings at finite temperature. Applying the rules of heterotic/type I duality, we inferred novel contributions to the free energy of a gas of type I superstrings. These contributions are due to BPS D-strings wrapped on internal circles which become massless at special points in moduli space, enhance the gauge group, and lift flat directions. These conclusions are based on the S-dual heterotic picture at weak coupling. At finite temperature, the latter is a no-scale model i.e.a flat background where all supersymmetries are spontaneously broken at tree level.

We computed the one-loop free energy density on the heterotic side for $D \geq 4$ and found points in moduli space where all internal moduli are dynamically stabilized due to the cosmological evolution. Additionally, in $D \geq 5$, the evolution of the dilaton asymptotes to a constant value, while in $D = 4$, the dilaton turns out to have a logarithmically decreasing behavior.

Using the S-duality, this implies that for $D \geq 7$, all type I internal moduli can be stabilized at strong coupling. In $D = 6$, the S-duality maps the heterotic coupling into the type I volume modulus. As a result the only remaining flat direction in type I is the internal volume modulus, which asymptotes to a constant finite value, while the type I dilaton is stabilized at weak coupling. For the cases $D \leq 5$, all type I internal moduli can be stabilized at weak coupling. Furthermore in $D = 4$, the type I dilaton inherits the logarithmic behavior from the heterotic dilaton, while it asymptotes to a constant in higher dimensions. In all cases, the late time geometric evolution is identical to a radiation dominated evolution. Furthermore, all solutions are stable under small perturbations and are thus local attractors of the dynamics.

It is worth stressing that the effects of the massless BPS non-perturbative D-strings persist at weak coupling, as their masses are protected by supersymmetry. As a result, the stabilization in type I for $D \geq 7$ persists at weak coupling. Furthermore, taking these modes into account is not optional in phenomenologically motivated uses of the type I superstring. Actually, this is not the first time massless solitons play an essential role in weakly coupled theories. For instance, in type IIB compactifications on Calabi-Yau threefolds, the conifold singularities in the vector multiplets moduli spaces are explained by massless hypermultiplets realized by D3-branes wrapped on vanishing 3-cycles [57].

Realistic models should include also a spontaneous breaking of $\mathcal{N}_4 = 1$ supersymmetry at a scale M, before finite temperature T is switched on. In this case, the universe is attracted to a "radiation-like dominated era" [25–32]. This evolution is characterized by

[13]We have also investigated a third local attractor at the point $\mathcal{T} = \mathcal{U} = i/2$, $Y_8^{I\geq 10} = 0 = Y_9^{10,11}$, $Y_9^{12,\ldots,25} = 1/2$, which corresponds to the gauge enhancement $SU(2) \times SU(2) \times SO(32)$ and a stabilization of all internal moduli. Due to its similarity, we do not present its details here.

coherent motions of $e^{4\phi(t)}$ (where ϕ is the dilaton in four dimensions) and the modulus $M(t)$, both proportional to $T(t)$ such that Friedmann's equation is effectively that of a radiation dominated era, $H^2 \propto T^4$. The energy stored in the oscillations of the moduli around their minima is found to be dominated by the thermal energy and so the stabilization of the scalars is guaranteed. Moreover, infrared effects are expected to put a halt to the run away behavior of the string coupling and supersymmetry breaking scale. In particular, when $T(t)$ reaches the electroweak scale $M_{\rm EW}$, radiative corrections are not screened anymore by temperature effects and the electroweak breaking is expected to take place [58–63]. This should be accompanied by the stabilization of $M(t)$ around $M_{\rm EW}$ [64]. Clearly, it is of utmost importance to implement these effects in our cosmological set up since this would provide a precise context for addressing questions of dark matter, astroparticle physics and phenomenology. Additionally for $D = 4$, as well as $D = 5$, there is the possibility of large contributions coming from light NS5-brane states in the heterotic theory or D5-brane states in the type I theory which have not been taken into account yet. It is possible that these states can play a role in stabilizing the dilaton. To make progress in this direction, one may try to exploit heterotic/type II duality in $D = 4$ which is a strong-weak duality.

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A Thermal partition functions

Type I superstring. To study the canonical ensemble of a perfect gas of maximally supersymmetric open and closed superstrings, we compactify the type I theory on the Euclidean background $S^1(R_{\rm I0}) \times T^{D-1} \times \prod_{i=D}^{9} S^1(R_{{\rm I}i})$. Bosons (fermions) are imposed periodic (antiperiodic) boundary conditions along $S^1(R_{\rm I0})$, where $\hat\beta_{\rm I} = 2\pi R_{\rm I0}$ is the inverse temperature. The spatial torus T^{D-1} is considered in the large volume $\hat V_{\rm I}$ limit. Our aim is to compute the one-loop thermal partition function. The treatment of a generic Scherk-Schwarz compactification can be found in [65] and the case of present interest is reviewed in [66].

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In the closed string sector, the torus contribution is half that of type IIB,

$$
\begin{aligned}
\mathcal{T} =& \frac{\hat\beta_{\rm I}\hat V_{\rm I}}{(2\pi)^D}\,\frac{1}{2}\int_{\mathcal F}\frac{d^2\tau}{2\tau_2^{\frac{D}{2}+1}}\,\frac{1}{\eta^8\bar\eta^8}\sum_{\vec m,\vec n} q^{\frac14 \vec p_L^2}\bar q^{\frac14 \vec p_R^2}\sum_{n^0,\tilde m_0} e^{-\frac{\pi R_{\rm I0}}{\tau_2}\left|n^0\tau+\tilde m_0\right|^2}\\
&\times\frac12\sum_{a,b}(-)^{a+b+ab}\frac{\theta[^a_b]^4}{\eta^4}\,\frac12\sum_{\bar a,\bar b}(-)^{\bar a+\bar b+\bar a\bar b}\frac{\bar\theta[^{\bar a}_{\bar b}]^4}{\bar\eta^4}\,(-)^{\tilde m_0(a+\bar a)+n^0(b+\bar b)}\\
=& \frac{\hat\beta_{\rm I}\hat V_{\rm I}}{(2\pi)^D}\,\frac{1}{2}\int_{\mathcal F}\frac{d^2\tau}{2\tau_2^{\frac{D}{2}+1}}\,\frac{1}{\eta^8\bar\eta^8}\sum_{\vec m,\vec n} q^{\frac14 \vec p_L^2}\bar q^{\frac14 \vec p_R^2}\\
&\left\{\sum_{n^0\,{\rm even},\,\tilde m_0} e^{-\frac{\pi R_{\rm I0}^2}{\tau_2}|n^0\tau+\tilde m_0|^2}\left[(V_8\bar V_8+S_8\bar S_8)-(-1)^{\tilde m_0}(V_8\bar S_8+S_8\bar V_8)\right]\right.\\
&\left.+\sum_{n^0\,{\rm odd},\,\tilde m_0} e^{-\frac{\pi R_{\rm I0}^2}{\tau_2}|n^0\tau+\tilde m_0|^2}\left[(O_8\bar O_8+C_8\bar C_8)-(-1)^{\tilde m_0}(O_8\bar C_8+C_8\bar O_8)\right]\right\},
\end{aligned}
\tag{A.1}
$$

where $q=e^{2i\pi\tau}$ and $p_{L,Ri}=m_i/R_{{\rm I}i}\mp n^iR_{{\rm I}i}$. The above second expression involves SO(8) affine characters, where those associated to the vectorial and spinorial representations satisfy

$$
\frac{V_8}{\eta^8}=\frac{S_8}{\eta^8}=\sum_{A\ge0}s_A\,q^A. \tag{A.2}
$$

The Klein bottle amplitude $\mathcal{K}$ is obtained by keeping all characters of $\mathcal{T}$ which are invariant under left $\leftrightarrow$ right symmetry. Symmetrizing and antisymmetrizing the NS-NS and RR sectors respectively, $\mathcal{K}$ involves the combination V_8-S_8 and is thus vanishing. In the open string sector, the thermal annulus and Möbius strip amplitudes are

$$
\mathcal{A} = \frac{\hat\beta_{\rm I}\hat V_{\rm I}}{(2\pi)^D}\,\frac{N^2}{2}\int_0^{+\infty}\frac{d\tau_2}{2\tau_2^{\frac{D}{2}+1}}\,\frac{1}{\eta^8}\sum_{\vec m}q^{\vec p^2}\sum_{\tilde m_0}e^{-\frac{\pi R_{\rm I0}^2}{\tau_2}\tilde m_0^2}\left[V_8-(-1)^{\tilde m_0}S_8\right], \tag{A.3}
$$

$$
\mathcal{M} = -\frac{\hat\beta_{\rm I}\hat V_{\rm I}}{(2\pi)^D}\,\frac{N}{2}\int_0^{+\infty}\frac{d\tau_2}{2\tau_2^{\frac{D}{2}+1}}\,\frac{1}{\hat\eta^8}\sum_{\vec m}q^{\vec p^2}\sum_{\tilde m_0}e^{-\frac{\pi R_{\rm I0}^2}{\tau_2}\tilde m_0^2}\left[\hat V_8-(-1)^{\tilde m_0}\hat S_8\right], \tag{A.4}
$$

where $N=32$, $q=e^{-\pi\tau_2}$, $p_i=m_i/R_{{\rm I}i}$ and the "hatted" characters in eq. (A.4) have the power expansion

$$
\frac{\hat V_8}{\hat\eta^8}=\frac{\hat S_8}{\hat\eta^8}=\sum_{A\ge0}(-)^A s_A\,q^A. \tag{A.5}
$$

We proceed by evaluating more explicitly the amplitude $\mathcal{T}$ by "unfolding" the fundamental domain of integration [67, 68]. In fact, for any set of modular covariant functions $f_{(n,\tilde m)}(\tau,\bar\tau)$ such that $f_{(n,\tilde m)}(M(\tau),M(\bar\tau))=f_{(n,\tilde m)M}(\tau,\bar\tau)$ for all $M\in{\rm SL}(2,\mathbb{Z})$, one has[14]

$$
\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\sum_{n,\tilde m}f_{(n,\tilde m)}(\tau,\bar\tau)=\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\,f_{(0,0)}(\tau,\bar\tau)+\int_{\mathcal{S}_+}\frac{d^2\tau}{\tau_2^2}\sum_{\tilde m\neq0}f_{(0,\tilde m)}(\tau,\bar\tau), \tag{A.6}
$$

[14]Eq. (A.6) is true as long as it is allowed to exchange discrete sum and integration, a fact which is guaranteed if the integrand is absolutely convergent. This condition is satisfied for $\mathcal{T}$ when $R_{\rm I0}>R_{\rm IH}$.

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where $\mathcal{S}_+$ is the upper half strip: $-1/2 < \tau_1 < 1/2$, $\tau_2 > 0$. Applied to eq. (A.1), supersymmetry implies that the contribution for $n_0 = \tilde{m}_0 = 0$ vanishes and we are left with integrals over $\mathcal{S}_+$ for $n_0 = 0$, $\tilde{m}_0 \neq 0$. Defining $\tilde{m}_0 = 2\tilde{k}_0 + 1$ and using (A.2), one obtains

$$\begin{aligned}
\mathcal{T} &= \frac{\hat{\beta}_{\rm I}\hat{V}_{\rm I}}{(2\pi)^D}\int_{\mathcal{S}_+}\frac{d^2\tau}{\tau_2^{\frac{D}{2}+1}}\sum_{\substack{\tilde{k}_0,\,\vec{m},\,\vec{n}\\ A\geq 0,\,\bar{A}\geq 0}} s_A s_{\bar{A}}\, e^{2i\pi\tau_1(A-\bar{A}-\vec{m}\cdot\vec{n})}\, e^{-\frac{\pi R_{\rm I0}^2}{\tau_2}(2\tilde{k}_0+1)^2-\pi\tau_2\left[2(A+\bar{A})+\sum_i\left(\frac{m_i^2}{R_{{\rm I}i}^2}+n^{i\,2}R_{{\rm I}i}^2\right)\right]}\\
&= \frac{\hat{\beta}_{\rm I}\hat{V}_{\rm I}}{(2\pi)^D}\int_0^{+\infty}\frac{d\tau_2}{\tau_2^{\frac{D}{2}+1}}\sum_{\substack{\tilde{k}_0,\,\vec{m},\,\vec{n}\\ A\geq 0,\,\bar{A}\geq 0\\ A-\bar{A}=\vec{m}\cdot\vec{n}}} s_A s_{\bar{A}} e^{-\frac{\pi R_{\rm I0}^2}{\tau_2}(2\tilde{k}_0+1)^2-\pi\tau_2\left[4A+\sum_i\left(\frac{m_i}{R_{{\rm I}i}}-n^iR_{{\rm I}i}\right)^2\right]},
\end{aligned}\tag{A.7}$$

where level matching is implemented by integrating over τ_1. Using the formula $\int_0^\infty dx\frac{e^{-a/x-bx}}{x^\nu} = 2a^{\frac{1-\nu}{2}}b^{\frac{\nu-1}{2}}K_{\nu-1}(2\sqrt{ab})$, where $K_\nu(x)$ is the modified Bessel function of second kind, the integral over τ_2 yields eqs. (2.1) and (2.2). Similarly, applying the expansions (A.2) and (A.5) in eqs. (A.3) and (A.4), we have

$$\mathcal{A} = \frac{\hat{\beta}_{\rm I}\hat{V}_{\rm I}}{(2\pi)^D}\frac{N^2}{2}\int_0^{+\infty}\frac{d\tau_2}{\tau_2^{\frac{D}{2}+1}}\sum_{\tilde{k}_0,\,\vec{m},\,A\geq 0} s_A\, e^{-\frac{\pi R_{\rm I0}^2}{\tau_2}(2\tilde{k}_0+1)^2-\pi\tau_2\left(\sum_i\frac{m_i^2}{R_{{\rm I}i}^2}+A\right)},\tag{A.8}$$

$$\mathcal{M} = -\frac{\hat{\beta}_{\rm I}\hat{V}_{\rm I}}{(2\pi)^D}\frac{N}{2}\int_0^{+\infty}\frac{d\tau_2}{\tau_2^{\frac{D}{2}+1}}\sum_{\tilde{k}_0,\,\vec{m},\,A\geq 0}(-)^A s_A\, e^{-\frac{\pi R_{\rm I0}^2}{\tau_2}(2\tilde{k}_0+1)^2-\pi\tau_2\left(\sum_i\frac{m_i^2}{R_{{\rm I}i}^2}+A\right)},\tag{A.9}$$

which gives eq. (2.3) after integration over τ_2.

Dual heterotic string. We proceed by deriving the partition function of the dual heterotic theory, which is compactified on $S^1(R_{\rm h0})\times T^{D-1}\times\prod_{i=D}^9 S^1(R_{{\rm h}i})$. Bosons and fermions are again given periodic and antiperiodic boundary conditions along the Euclidean time circle, whose circumference defines the inverse temperature $\hat{\beta}_{\rm h} = 2\pi R_{\rm h0}$. This yields

$$\begin{aligned}
Z_{\rm h} &= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^D}\int_{\mathcal{F}}\frac{d^2\tau}{2\tau_2^{\frac{D}{2}+1}}\frac{\Gamma_{(0,16)}}{\eta^8\bar{\eta}^{24}}\sum_{\vec{m},\vec{n}} q^{\frac{1}{4}\vec{p}_L^2}\bar{q}^{\frac{1}{4}\vec{p}_R^2}\\
&\quad\times\sum_{n^0,\tilde{m}_0} e^{-\frac{\pi R_{\rm h0}^2}{\tau_2}|n^0\tau+\tilde{m}_0|^2}\,\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\frac{\theta[^a_b]}{\eta^4}\,(-)^{\tilde{m}_0a+n^0b+\tilde{m}_0n^0}\\
&= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^D}\int_{\mathcal{F}}\frac{d^2\tau}{2\tau_2^{\frac{D}{2}+1}}\frac{\Gamma_{(0,16)}}{\eta^8\bar{\eta}^{24}}\sum_{\vec{m},\vec{n}} q^{\frac{1}{4}\vec{p}_L^2}\bar{q}^{\frac{1}{4}\vec{p}_R^2}\left\{\sum_{n^0\text{ even},\,\tilde{m}_0} e^{-\frac{\pi R_{\rm h0}}{\tau_2}|n^0\tau+\tilde{m}_0|^2}\left[V_8-(-1)^{\tilde{m}_0}S_8\right]\right.\\
&\quad\left.+\sum_{n^0\text{ odd},\,\tilde{m}_0} e^{-\frac{\pi R_{\rm h0}}{\tau_2}|n^0\tau+\tilde{m}_0|^2}\left[(-1)^{\tilde{m}_0}O_8-C_8\right]\right\},
\end{aligned}\tag{A.10}$$

where $q = e^{2i\pi\tau}$, while $p_{L,Ri} = m_i/R_{{\rm h}i}\mp n^iR_{{\rm h}i}$ and the volume $\hat{V}_{\rm h}$ are now measured in the heterotic theory. Alternatively, the lattice of internal zero modes can be considered in its

Lagrangian formulation, as needed in section 4 for the direction 9,

$$\sum_{m_9,n_9} q^{\frac{1}{4}p_{L9}^2}\bar{q}^{\frac{1}{4}p_{R9}^2} = \frac{R_{\rm h9}}{\sqrt{\tau_2}}\sum_{n^9,\tilde{m}_9} e^{-\frac{\pi R_{\rm h9}^2}{\tau_2}|n^9\tau+\tilde{m}_9|}. \tag{A.11}$$

To unfold the fundamental domain of integration in (A.10), one can use the identity (A.6) as in the torus amplitude in type I. Expanding the SO(32) right-moving lattice as

$$\frac{\Gamma_{(0,16)}}{\bar{\eta}^{24}} = \sum_{\bar{A}\geq -1} b_{\bar{A}}\,\bar{q}^{\bar{A}}, \tag{A.12}$$

and using eq. (A.2), one obtains

$$\begin{aligned} Z_{\rm h} &= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^D}\int_{\mathcal{S}_+}\frac{d^2\tau}{\tau_2^{\frac{D}{2}+1}}\sum_{\substack{\tilde{k}_0,\,\vec{m},\,\vec{n}\\ A\geq 0,\,\bar{A}\geq -1}} s_A b_{\bar{A}}\, e^{2i\pi\tau_1(A-\bar{A}-\vec{m}\cdot\vec{n})}\, e^{-\frac{\pi R_{\rm h0}^2}{\tau_2}(2\tilde{k}_0+1)^2-\pi\tau_2\left[2(A+\bar{A})+\sum_i\left(\frac{m_i^2}{R_{\rm hi}^2}+n^{i\,2}R_{\rm hi}^2\right)\right]} \\ &= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^D}\int_0^\infty\frac{d\tau_2}{\tau_2^{\frac{D}{2}+1}}\sum_{\substack{\tilde{k}_0,\,\vec{m},\,\vec{n}\\ A\geq 0,\,\bar{A}\geq -1\\ A-\bar{A}=\vec{m}\cdot\vec{n}}} s_A b_{\bar{A}}\, e^{-\frac{\pi R_{\rm h0}^2}{\tau_2}(2\tilde{k}_0+1)^2-\pi\tau_2\left[4A+\sum_i\left(\frac{m_i}{R_{\rm hi}}-n^iR_{\rm hi}\right)^2\right]}, \end{aligned} \tag{A.13}$$

which can be integrated to give eqs. (3.1) and (2.2).

Heterotic string at generic point in moduli space. In sections 5 and 6 for $D=8$, we study in the context of the maximally supersymmetric heterotic string the stabilization of all internal moduli by the free energy density at weak coupling. In Einstein frame, the latter is $\mathcal{F} = -e^{\frac{2D}{D-2}\phi}Z_{\rm h}/(\hat{\beta}_{\rm h}\hat{V}_{\rm h})$, where ϕ is the dilaton in dimension D and $Z_{\rm h}$ is the vacuum energy in the Euclidean background $S^1(R_{\rm h0})\times T^{D-1}\times T^{10-D}$. The internal moduli are the metric $\hat{g}_{ij}$, the antisymmetric tensor B_{ij} and the Wilson lines Y_i^I ($i,j=D,\dots,9$; $I=10,11,\dots,25$). Proceeding as before, the partition function $Z_{\rm h}$ takes the following forms,

$$\begin{aligned} Z_{\rm h} &= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^D}\int_{\mathcal{F}}\frac{d^2\tau}{2\tau_2^{\frac{D}{2}+1}}\sum_{\vec{m},\vec{n},\vec{Q}}\frac{q^{\frac{1}{4}\vec{p}_L^2}\bar{q}^{\frac{1}{4}\vec{p}_R^2}}{\eta^8\bar{\eta}^{24}}\sum_{n^0,\tilde{m}_0} e^{-\frac{\pi R_{\rm h0}^2}{\tau_2}|n^0\tau+\tilde{m}_0|^2}\,\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\frac{\theta[^a_b]}{\eta^4}\,(-)^{\tilde{m}_0a+n^0b+\tilde{m}_0n^0} \\ &= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^D}\int_{\mathcal{S}_+}\frac{d^2\tau}{2\tau_2^{\frac{D}{2}+1}}\sum_{\vec{m},\vec{n},\vec{Q}} q^{\frac{1}{4}\vec{p}_L^2}\bar{q}^{\frac{1}{4}\vec{p}_R^2}\sum_{\tilde{k}} e^{-\frac{\pi R_{\rm h0}^2}{\tau_2}(2\tilde{k}_0+1)^2}\,\frac{V_8+S_8}{\eta^8\bar{\eta}^{24}} \\ &= \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^D}\int_{\mathcal{S}_+}\frac{d^2\tau}{\tau_2^{\frac{D}{2}+1}}\sum_{\substack{\tilde{k}_0,\,\vec{m},\,\vec{n},\,\vec{Q}\\ A\geq 0,\,\bar{A}\geq -1}} s_A r_{\bar{A}}\, e^{2i\pi\tau_1\left(A-\bar{A}+\frac{1}{4}(\vec{p}_L^2-\vec{p}_R^2)\right)}\, e^{-\frac{\pi R_{\rm h0}^2}{\tau_2}(2\tilde{k}_0+1)^2-\pi\tau_2\left[2(A+\bar{A})+\frac{1}{2}(\vec{p}_L^2+\vec{p}_R^2)\right]}, \end{aligned} \tag{A.14}$$

where we introduce the coefficients $r_{\bar{A}}$ of the expansion $\bar{\eta}^{-24} = \sum_{\bar{A}\geq -1} r_{\bar{A}}\bar{q}^{\bar{A}}$. The moduli-dependent internal momenta are specified by $\vec{m}$, $\vec{n}$ and the root vector Q^I of the right-moving lattice $\Gamma_{Spin(32)/\mathbb{Z}_2}$ [69, 70],

$$\begin{aligned} p_{L,R}^I &= \left(m_i - Q^JY_i^J - n^jB_{ij} - \frac{1}{2}n^jY_i^JY_j^J\right)e^{*iI} \mp n^ie_i^I \ \text{ for } i,j,I=D,\dots,9;\ J=10,\dots,25, \\ p_R^I &= \sqrt{2}\left(Q^I + n^iY_i^I\right) \ \text{ for } I=10,\dots,25;\ \vec{Q}\in\Gamma_{\rm Spin(32)/\mathbb{Z}}, \end{aligned} \tag{A.15}$$

where $\{e_i\}$ is a vector basis of T^{10-D} i.e.$\hat{g}_{ij} = e_i^I e_j^I$ and $e^{*iI} e_j^I = \delta_j^i$. Since these momenta satisfy $\frac{1}{2}(\vec{p}_L^2 - \vec{p}_R^2) = -2\vec{m}\cdot\vec{n} - \vec{Q}\cdot\vec{Q}$, the level matching condition implemented by integrating over τ_1 in eq. (A.14) is $A - \bar{A} = \vec{m}\cdot\vec{n} + \frac{1}{2}\vec{Q}\cdot\vec{Q}$, which yields

$$Z_{\rm h} = \frac{\hat{\beta}_{\rm h}\hat{V}_{\rm h}}{(2\pi)^D} \int_0^\infty \frac{d\tau_2}{\tau_2^{\frac{D}{2}+1}} \sum_{\substack{\tilde{k}_0,\, \vec{m},\, \vec{n},\, \vec{Q} \\ A\geq 0,\, \bar{A}\geq -1 \\ A-\bar{A}=\vec{m}\cdot\vec{n}+\frac{1}{2}\vec{Q}\cdot\vec{Q}}} s_A r_{\bar{A}}\, e^{-\frac{\pi R_0^2}{\tau_2}(2\tilde{k}_0+1)^2 - \pi\tau_2 \hat{M}^2_{A,\vec{m},\vec{n},\vec{Q}}(\hat{g},B,Y)}, \tag{A.16}$$

where $\hat{M}^2_{A,\vec{m},\vec{n},\vec{Q}}(\hat{g},B,Y) = 2(A+\bar{A}) + \frac{1}{2}\left(\vec{p}_L^2 + \vec{p}_R^2\right)$ are the masses of the boson/fermion pairs of superpartners. Integrating over τ_2, the above expression for Z_h leads to the free energy density (5.9), while for $D = 8$ the mass spectrum takes the more explicit form (6.5).

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MODULI STABILIZATION IN EARLY SUPERSTRING COSMOLOGY

Lihui Liu

Centre de Physique Théorique, Ecole Polytechnique[†]
F–91128 Palaiseau cedex, France
lihui.liu@cpht.polytechnique.fr

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Abstract

We study moduli stabilization by thermal effects in the cosmological context. The implementation of finite temperature, which spontaneously breaks supersymmetry, induces an effective potential at one loop level. At the points where extra massless states appear in the string spectrum, the potential develops local minima whose depth depends on the temperature. Moduli attracted to these points acquire dynamical masses which decrease with cosmological evolution. This makes the coherent scalar oscillations dilute before nucleosynthesis, and the cosmological moduli problem is avoided. In particular, we study the effective potential induced by a maximally supersymmetric heterotic string gas for spacetime dimension $D \geq 4$, and a gas of type II strings compactified on Calabi-Yau three-folds ($D = 4$). In the former case, the local minima of the potential arise at enhanced gauge symmetry points, which can stabilize all moduli but the dilaton. In the latter case, the local minima are reached at the loci where 2-cycles or 3-cycles in the Calabi-Yau space shrink to zero size, accompanied with either conifold transitions or non Abelian gauge symmetries. This stabilizes the type II moduli which characterize the deformation of these shrinking cycles. Moduli stabilization in the dual string models is also investigated by heterotic/type I S-dualities and type II/heterotic dualities.

[†] Unité mixte du CNRS et de l'Ecole Polytechnique, UMR 7644.

1 Introduction and outline

Moduli stabilization is a long standing issue in the superstring phenomenology. In fact, the presence of moduli fields in supersymmetric compactifications leads to difficulties: not only the massless scalars are in contradiction with observations of the gravitational force, but also, being continuous parameters in the couplings and mass spectrum, they imply a loss of predictability of the theory. Much attention is thus drawn to the search for mechanisms which attract the moduli fields to certain preferred values, where scalar masses are generated.

On the other hand, the scalar masses are subjected to constraints from cosmology. Basically, when the scalar fields oscillate coherently in the potential well, the energy of oscillation dominates the total energy of the universe [1], until the scalar particles decay. The productions of the decay can alter the primordial abundances of the light nuclei produced by nucleosynthesis. Also, the huge amount of entropy production during the decay can wash out the baryon number asymmetry. This is termed as the cosmological moduli problem, which was initially identified in the framework of supersymmetric standard models [2–4]. One applaudable solution to these is to require the scalar masses be of $O(10)$TeV order. It is pointed out in [3] that once this is satisfied, the decay of these scalar particles reheats the universe to a temperature of order 1MeV, high enough to restart the nucleosynthesis. Then it is found in [4] that the baryon number asymmetry can also be saved by the $O(10)$TeV order scalar mass if the baryogenesis is due to the Affleck-Dine mechanism [5].

Here I present our recent work [6, 7] where the above problems were addressed by investigating thermal string effects. It was shown in [8] that a gas of string modes, carrying both winding and momenta, can generate a free energy that enables stabilization of radii moduli. A quantum version of this effect has been presented in [6, 7, 9, 10], with the thermal gas and free energy replaced by virtual strings which induce an effective potential. To avoid generating a large cosmological constant, the cosmology is addressed in the context of no-scale models [11]. The latter are defined at classical level by backgrounds associated to vanishing minima of the scalar potential, with flat directions parameterized by the spontaneous supersymmetry breaking scale.

For simplicity, we consider here only temperature breaking of supersymmetry. At the level of conformal field theory on the worldsheet, the implementation of finite temperature amounts to a Scherk-Schwarz reduction on the Euclidean time circle of radius R_0, with boundary conditions associated to the spacetime fermion number [12]. The string frame temperature is $\hat{T} = \hat{\beta}^{-1} = 1/2\pi R_0$ and the Einstein frame temperature is $T = e^{\frac{2}{D-2}\phi^{(D)}}\,\hat{T}$, where $\phi^{(D)}$ is the D-dimensional dilaton. The supersymmetry is thus broken spontaneously at the scale T. We restrict our attention to the intermediate era between the Hagedorn phase transition [13] and the electroweak phase transition, so that $M_{\text{string}} \gg T \gg \Lambda_{\text{EW}}$.

To build phenomenologically viable models however, it is necessary to also include zero temperature spontaneous supersymmetry breaking. Otherwise as the temperature drops during the cosmological evolution, the supersymmetry broken by temperature will be restored. The case with another Scherk-Schwarz reduction performed in one of the internal dimensions is intensively studied in Refs. [9], where it is shown that the supersymmetry breaking scale M_{SUSY} induced in this internal dimension evolves proportionally with T. It is expected that by the end of the intermediate era, when T approaches $\Lambda_{\rm EW}$, the radiative corrections induced by infrared effects start to destabilize the Higgs potential, freezing $M_{\rm SUSY}$ at about $O(1)$TeV order. This gives an account of the hierarchy $M_{\rm SUSY} \ll M_{\rm Plank}$.

The breaking of supersymmetry generates a nontrivial vacuum-vacuum amplitude, which we compute at one-loop level, supposing that the string theory is at weak coupling. This amplitude is just the thermal partition function of the string gas computed up to one-loop level, which we denote by Z. It gives rise to the free energy density by $\mathcal{F} = -\frac{Z}{\beta V}$, ($V$ the space volume in Einstein frame). The back-reaction of $\mathcal{F}$ on the spacetime background is dictated by the one-loop effective action

$$S = \int d^D x \sqrt{-g}\Big[\frac{R}{2} - \frac{1}{2} F_{MN}\partial\Phi^M\partial\Phi^N - \mathcal{F}\big(T, \vec{\Phi}\big)\Big], \tag{1.1}$$

where Φ^M are the moduli, and the metric components F_{MN} are functions of these moduli. Since $\mathcal{F}$ appears in the action as the effective potential, moduli attractors should be its local minima.

It can be shown that when we only have temperature breaking of supersymmetry, the free energy density takes the form [6]:

$$\mathcal{F}(T,\vec{\Phi}) = -\int_0^\infty \frac{d\ell}{2\ell}\frac{1}{(2\pi\ell)^{\frac{D}{2}}}\sum_s e^{-\frac{1}{2}M_s(\vec{\Phi})^2\ell}\sum_{k_0\in\mathbb{Z}} e^{-\frac{(2k_0+1)^2}{2T^2\ell}} = -T^D\sum_s G\big(M_s(\vec{\Phi})/T\big), \tag{1.2}$$

where M_s is the tree-level mass of the s-th string state, which can depend on the moduli. The function $G(x)$ is defined in terms of the modified Bessel function of the second kind (see [6] for more details). It peaks at $x = 0$ and is exponentially suppressed at large x. Therefore only light states give significant contribution to $\mathcal{F}$. By consequence the local minima of $\mathcal{F}$ appear at the vacuum expectation values (VEV's) of $\vec{\Phi}$ where some massive states in the string spectrum become massless. These states can originate either from the perturbative spectrum or from non perturbative objects such as D-branes. String-string dualities can help figure out the non perturbative contribution.

The local minima of $\mathcal{F}$ induce time-dependent scalar masses, instead of constant ones. This ensures that the universe is radiation dominated at the exit of the intermediate era, which is crucial to the resolution of the cosmological moduli problem. In order to show this, we take the flat Robinson-Walker metric $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$ (in the Einstein frame). Solving the equations of motion about a local minimum of $\mathcal{F}$, say $\vec{\Phi}_0$, we obtain the following time evolution of the scale factor, the temperature, and the total energy density:

$$a(t) \propto 1/T(t) \propto t^{2/D}, \quad \rho_{\rm tot} \propto H^2 \propto a^{-D}. \tag{1.3}$$

The coherent moduli field oscillations obey the equation

$$\ddot{\epsilon}^M + (D-1)H\dot{\epsilon}^M + \Lambda^M_N\epsilon^N = 0, \tag{1.4}$$

where we let $\vec{\Phi} = \vec{\Phi}_0 + \vec{\epsilon}$, and we have the squared-mass matrix $\Lambda^M_N = \big(F^{MP}\mathcal{F}_{PN}\big)_{\vec{\Phi}_0}$, with $\mathcal{F}_{PN} := \partial^2\mathcal{F}\big/\partial\Phi^P\Phi^N$, and F^{MN} the inverse of F_{MN}. Using Eq.(1.2) one can show that

$\Lambda^M_N \propto T^{D-2}$. Thus with Eq.(1.4) we have the usual scalar dynamics, but with scalar masses depending on the temperature, hence on time. This results in the late-time scalar oscillation behavior $\epsilon \sim t^{-1/2} \sin\big(\lambda\, t^{2/D} + \text{phase}\big)$, instead of $t^{1/D-1} \sin\big(\lambda\, t + \text{phase}\big)$ for constant mass, where λ^2 is some eigenvalue of the squared-mass matrix. Therefore the energy density stored in the scalar oscillations behaves asymptotically as

$$\rho_\Phi = \frac{1}{2} F_{MN}\big|_{\vec{\Phi}_0}\, \dot{\epsilon}^M \dot{\epsilon}^N \sim t^{\frac{4}{D}-3} \propto a^{2-3D/2}, \tag{1.5}$$

while the result for the case of constant scalar mass is $\rho_\Phi \sim a^{1-D}$ which obviously dominates over the radiation energy $\rho_{\rm rad} \sim a^{-D}$ for any spacetime dimension. Back to the case of dynamical mass, where Eq.(1.5) holds, when $D \geq 5$, the universe is radiation dominated, since compared to Eq.(1.3), we have $\rho_\Phi \ll \rho_{\rm tot}$. However $D = 4$ is a marginal case where even though the metric evolution appears as that of a radiation dominated universe ($\rho_{\rm tot} \propto H^2 \propto a^{-4}$), the energy of coherent scalar oscillations is not dominated. Instead, it takes up a constant portion in the total energy ($\rho_{\rm tot} \propto \rho_\Phi \propto a^{-4}$). This is due to the over-simplified supersymmetry breaking mechanism that we adopt here. It is shown in [9] that the $D = 4$ case is also radiation dominated once an additional source of spontaneous supersymmetry breaking is introduced in the internal space.

In the following, we investigate the cosmology induced by two specific string models: the maximally supersymmetric heterotic strings and the Calabi-Yau (CY) compactification of type II strings.

2 Heterotic cosmology and type I dual

We start with the cosmology induced by weakly coupled $SO(32)$ heterotic strings compactified on a factorized torus $\prod_{i=D}^{9} S^1(R_{\text{h}i})$, where the subscript h indicates heterotic quantities. The model have maximal number of supersymmetry, so that the metric (F_{MN}) in Eq.(1.1) is exact at tree level. Let the moduli space be coordinatized by the D-dimensional dilaton $\phi_{\text{h}}^{(D)} := \phi_{\text{h}}^{(10)} - \frac{1}{2}\sum_{i=D}^{9} \ln(2\pi R_{\text{h}i})$ and all the internal radii $R_{\text{h}i}$ with $i = D, \dots, 9$. Computing the thermal one-loop amplitude, we find that when all radii satisfy $|R_{\text{h}i} - 1/R_{\text{h}i}| < 1/(2\pi R_{\text{h}0})$, $i = D, ..., 9$, the corresponding free energy density takes the form [6]:

$$\mathcal{F}_{\text{h}} = -T^D \left\{ n_0\, c_D + \sum_{i=D}^{9} n_1\, G\bigg(2\pi R_{\text{h}0}\Big|\frac{1}{R_{\text{h}i}} - R_{\text{h}i}\Big|\bigg) + \mathcal{O}(e^{-2\pi R_{\text{h}0}}) \right\}, \tag{2.1}$$

where the coefficients n_0 and n_1 are positive, associated to the counting of states. The first term in the above expression is from massless states. The second term involving the G-function shows that $\mathcal{F}_{\text{h}}$ reaches a local minimum at the self T-dual point $R_{\text{h}i} = 1$ ($i = D, \dots, 9$), due to the states of masses $\big|\frac{1}{R_{\text{h}i}} - R_{\text{h}i}\big|$. These are just the non Cartan components responsible for the gauge symmetry enhancement $U(1) \to SU(2)$ in each internal circle. In fact in heterotic strings, the correspondence between the enhancement of gauge symmetry and the local extrema of the free energy is true to all loop levels [14]. Therefore the internal radii can all be stabilized at the value 1 where we have $SU(2)^{10-D}$ enhanced symmetry.

Moreover for $D \geq 5$, the string coupling $\lambda_{\rm h}^{(D)} = e^{\phi_{\rm h}^{(D)}}$ freezes on the flat direction to some constant value determined by the initial conditions. For $D = 4$, the dilaton $\phi_{\rm h}^{(4)}$ does not converge to a constant but instead decreases logarithmically with the cosmological time.

We switch to the dual type I picture. If we perform naive perturbative computation $Z_{\rm I} = \mathcal{T} + \mathcal{K} + \mathcal{A} + \mathcal{M}$ to obtain the free energy density, we will find no local minimum of $\mathcal{F}_{\rm I}$, since no perturbative effect can lead to gauge symmetry enhancements in maximally supersymmetric type I strings. We thus seek to include non perturbative effects which can be inferred from heterotic strings through string-string S-dualities. In dimension D, the duality dictionary for Einstein frame quantities is [15]

$$\begin{aligned} R_{{\rm h}i} &= \frac{R_{{\rm I}i}}{\sqrt{\lambda_{\rm I}}} \equiv R_{{\rm I}i}\, \frac{e^{-\frac{1}{2}\phi_{\rm I}^{(D)}}}{\left(\prod_{j=D}^{9} 2\pi R_{{\rm I}j}\right)^{1/4}}\,, \qquad i = 0 \text{ or } D, ..., 9, \\ \phi_{\rm h}^{(D)} &= -\frac{D-6}{4}\,\phi_{\rm I}^{(D)} - \frac{D-2}{8}\sum_{i=D}^{9} \ln\left(2\pi R_{{\rm I}i}\right), \end{aligned} \tag{2.2}$$

where $\lambda_{\rm I}$ is the type I string coupling in ten dimensions. When applying this duality map, the heterotic states that induce the local minimum in Eq.(2.1) are sent to non perturbative states of masses $\left|\frac{1}{R_{{\rm h}i}} - \frac{R_{{\rm h}i}}{\lambda_{\rm I}}\right|$ on the type I side. From the type I point of view, they have the natural interpretation as D (or anti-D)-strings wrapped once along the circles $S^1(R_{{\rm I}i})$, with one unit of momentum. Therefore when all radii satisfy $\left|\frac{1}{R_{{\rm I}i}} - \frac{R_{{\rm I}i}}{\lambda_{\rm I}}\right| < \frac{1}{2\pi R_{\rm I0}}$, they are attracted to $R_{{\rm I}i} = \sqrt{\lambda_{\rm I}}$, where we have the enhanced gauge symmetry $SU(2)^{10-D}$ due to D-string states. The type I dilaton freezes somewhere along its flat direction just as its heterotic dual except for $D = 6$ where it is stabilized while the internal space volume $\prod_{i=D}^{9}(2\pi R_{{\rm I}i})$ freezes along a flat direction. This is because in $D = 6$ the duality map Eq.(2.2) exchanges internal volumes and string couplings. Another subtlety arising from Eq.(2.2) is that, since the heterotic theory is always in the weak coupling regime, the type I dual is strongly coupled for $D > 6$ and weakly coupled for $D < 6$. However our result is still valid at small coupling for $D > 6$ since the D-string states, responsible for the stabilization of $R_{{\rm I}i}$, are BPS states whose masses are protected by supersymmetry.

The D-string state contribution can also have an E1-instanton interpretation, following the lines of Refs. [16]. For simplicity, we consider the compactification on $S^1(R_{\rm I9})$. This contrasts the zero temperature case where E1-instantons arise for $D \leq 8$. Starting from the heterotic side, we can easily express the thermal partition function as a sum over world-sheet instantons. When sending this heterotic result to the type I side using the dictionary (2.2), the corresponding type I partition function contains a sum of E1-instantons, which is explicitly [6]

$$Z_{\rm I}^{E1} = \frac{\hat{V}_{\rm I}^{(10)}}{(2\pi)^{10}}\, 2 \sum_{\text{E1 instantons}} s_0\, \frac{e^{\frac{2i\pi}{\lambda_{\rm I}}\Upsilon_{\rm I}}}{\Upsilon_{\rm I2}\, \mathcal{Y}_{\rm I2}^4} \sum_{n=0}^{4} \left[\frac{\alpha_n}{(2\pi\Upsilon_{\rm I2})^n} \sum_{\bar{A}\geq -1} b_{\bar{A}} \left(\frac{1}{\lambda_{\rm I}} + \bar{A}\,\frac{\mathcal{Y}_{\rm I2}}{\Upsilon_{\rm I2}}\right)^{4-n} e^{2i\pi \mathcal{Y}_{\rm I}\bar{A}}\right] + c.c. + \mathcal{O}(e^{-4\pi\frac{R_{\rm I0}}{\sqrt{\lambda_{\rm I}}}}), \tag{2.3}$$

with the Kähler and complex structure moduli $\Upsilon_{\rm I}$ and $\mathcal{Y}_{\rm I}$ of the torus $S^1(R_{\rm I0})\times S^1(R_{\rm I9})$

$$\left\{\begin{array}{l}\Upsilon_{\rm I}=i\Upsilon_{\rm I2}=i(2\tilde{k}_0+1)R_{\rm I0}\cdot n^9R_{\rm I9}\\ \mathcal{Y}_{\rm I}=\mathcal{Y}_{\rm I1}+i\mathcal{Y}_{\rm I2}=\dfrac{\tilde{m}_9}{n^9}+i\,\dfrac{(2\tilde{k}_0+1)R_{\rm I0}}{n^9R_{\rm I9}}\end{array}\right. ,\qquad n^9>\tilde{m}_9\geq 0\ ,\ \ \tilde{k}_0\geq 0. \tag{2.4}$$

This result suggests it possible to derive from a pure type I point of view the free energy responsible for the stabilization of the internal moduli.

We can further consider generic toroidal compactifications, where all possible moduli are switched on. On the heterotic side, these moduli include the dilaton $\phi_{\rm h}^{(D)}$, the internal metric $g_{ij}^{\rm (h)}$, the internal antisymmetric tensor $B_{ij}^{\rm (h)}$, and the Wilson lines $Y^I_{{\rm (h)}i}$, where $i,j=D,\dots,9$ and $I=10,\dots,25$. Again, all moduli except the dilaton are attracted to the values associated to some enhanced gauge symmetry, where $\mathcal{F}_{\rm h}$ is minimized locally. In the dual type I picture, moduli stabilization is inferred from the heterotic side through the dictionary

$$\begin{aligned}\phi_{\rm h}^{(D)}&=-\frac{D-6}{4}\phi_{\rm I}^{(D)}-\frac{D-2}{8}\ln\sqrt{g^{(h)}},\\ g_{ij}^{\rm (h)}&=\frac{g_{ij}^{\rm (I)}}{\lambda_{\rm I}},\quad B_{ij}^{\rm (h)}=C_{ij},\quad Y^I_{{\rm (h)}i}=Y^I_{{\rm (I)}i}.\end{aligned} \tag{2.5}$$

where C_{ij} is the Ramond-Ramond 2-form. The subtlety is that now the dual type I moduli are stabilized by either non perturbative D-string states in the closed string sector or perturbative states in the open string sector. For $D\neq 6$, all type I moduli are stabilized except the dilaton, and for $D=6$ however, the dilaton is stabilized while the internal volume freezes on a flat direction.

As an explicit example, we examine the case of compactification on T^2, where we have on the heterotic side, the moduli $\mathcal{T}=B_{89}+i\sqrt{\hat{g}_{88}\hat{g}_{99}-\hat{g}_{89}^2}$, $\mathcal{U}=\big(\hat{g}_{89}+i\sqrt{\hat{g}_{88}\hat{g}_{99}-\hat{g}_{89}^2}\big)/\hat{g}_{88}$ and the Wilson lines Y_i^I ($i,j=8,9$; $I=10,11,\dots,25$). The mass formula for perturbative F-string states is

$$\hat{M}^2_{A,\vec{m},\vec{n},\vec{Q}}(\mathcal{T},\mathcal{U},Y)=\frac{1}{\mathcal{T}_2\mathcal{U}_2}\left|-m_8\mathcal{U}+m_9+\tilde{\mathcal{T}}n^8+\Big(\tilde{\mathcal{T}}\mathcal{U}-\frac{1}{2}\mathcal{W}^I\mathcal{W}^I\Big)n^9+\mathcal{W}^IQ^I\right|^2+4A, \tag{2.6}$$

where $\mathcal{W}^I:=\mathcal{U}Y_8^I-Y_9^I$ and $\tilde{\mathcal{T}}:=\mathcal{T}+\frac{1}{2}Y_8^I\mathcal{W}^I$, $\vec{m}$, $\vec{n}$ are the internal momenta and winding numbers, and Q^I the root vector of the internal lattice $\Gamma_{O(32)/\mathbb{Z}_2}$. Using the mass formula we can figure out moduli attractors where there are states becoming massless. The enhanced gauge group can be determined from the Narrain lattice formed by the right-moving internal momenta of these states. For example we have the local attractor with $SU(3)\times SO(32)$ enhanced symmetry, where the moduli are stabilized at $Y_i^I=0$, $\mathcal{T}=\mathcal{U}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Another less trivial example is the attractor with $SU(2)\times SO(34)$ enhanced symmetry, where the moduli are attracted to $\mathcal{T}=\mathcal{U}=i/\sqrt{2}$, $Y_8^{I\geq 10}=0$ and $Y_9^{10}=-Y_9^{11}=-Y_9^{12}=\cdots=-Y_9^{25}=-1/2$. In the dual type I picture the moduli stabilization follows from the dictionary (2.5).

3 Type II cosmology and heterotic dual

We turn to models with less supersymmetry. We consider cosmology in type II strings compactified on a Calabi-Yau (CY) three-fold M of Hodge numbers (h_{11}, h_{12}). The moduli space is a Cartesian product $\mathcal{M}_V \times \mathcal{M}_H$. The vector multiplet moduli space $\mathcal{M}_V$ of complex dimension h_{11} is a special Kähler manifold, which is exact at tree level, because the dilaton lives in a hypermultiplet. The hypermultiplet moduli space $\mathcal{M}_H$ of real dimension $4(h_{12}+1)$ is a quaternionic manifold, which contains the universal hypermultiplet accommodating the dilaton. Therefore $\mathcal{M}_H$ is subjected to perturbative and non perturbative corrections. When M is a $K3$ fibration, a dual heterotic string theory can exist, compactified on $K3 \times T^2$. The string-string duality sends the type II vector multiplet moduli to the heterotic vector multiplet moduli and the same is true of the hypermultiplet moduli. Thus the stabilization of heterotic moduli can be inferred from the stabilization of the dual type II moduli.

On the type II side, the moduli space develops singular loci when the internal CY space undergoes extremal transitions. At these loci, some 2-cycles or 3-cycles in the CY space shrink to zero size, giving rise to a singular three-fold. This can lead to conifold transitions or non Abelian gauge symmetries, with extra massless states appearing in the low-energy spectrum. Nonsingular CY three-folds can be recovered by restoring the shrinking 2-cycles or 3-cycles to finite size. In the following analysis, we adopt the type IIA description, and suppose that the desingularization by restoring 2-cycles is always available. Indeed only in this case can we write down the effective gauge theory, following the analysis in [17, 18].

At the conifold locus, let R 2-cycles in the CY space M, spanning an S-dimensional subspace of homology, shrink to separated nodes. Locally, R monopole states become massless, described by R hypermultiplets charged under S $U(1)$-vector multiplets [17]. When $R > S$, we can deform the shrinking 2-cycles into 3-cycles and obtain a topologically different CY space M'. The change in Hodge numbers is

$$h_{11}(M') = h_{11}(M) - S, \quad h_{12}(M') = h_{12}(M) + R - S. \tag{3.1}$$

Near the non Abelian locus, $N-1$ homologically independent 2-cycles, with the intersection matrix the Cartan matrix of A_{N-1}, shrink to zero size along a smooth curve C of genus g. By the arguments in [18], $N^2 - N$ vector multiplets and $g(N^2 - N)$ hypermultiplets become massless, giving rise to the gauge symmetry enhancement $U(1)^{N-1} \to SU(N)$ with g hypermultiplets transforming in its adjoint representation. When $g > 1$, we can construct a topologically different CY space M'' by deforming all shrinking 2-cycles into 3-cycles. The change in Hodge numbers is

$$h_{11}(M'') = h_{11}(M) - (N-1), \quad h_{12}(M'') = h_{12}(M) + (g-1)(N^2 - N) - (N-1). \tag{3.2}$$

In both cases, the low energy effective theory about the singular loci containing all light fields is described by a gauged $\mathcal{N}_4 = 2$ supergravity theory. Desingularizing the CY space by restoring the shrinking 2-cycles (3-cycles) corresponds to sitting in the Coulomb (Higgs) branch of the gauge theory. Therefore by our setup, the Coulomb branch must exist. The scalar fields in the light vector multiplets span a special Kähler manifold which contains $\mathcal{M}_V$, and we denote its special coordinates by $\{X^I\}$, $I = 1, \ldots, n_V$. The scalar fields in the light hypermultiplets span a quaternionic manifold which contains $\mathcal{M}_H$, and we let its real coordinates be $\{q^\Xi\}$, $\Xi = 1, \ldots, 4n_H$. Here $n_V \geq h_{11}$ and $n_H \geq h_{12} + 1$ are respectively the total number of light vector multiplets and light hypermultiplets. These scalar fields are divided into two groups: those participating in the extremal transition whose VEV's characterize the deformation of the vanishing cycles, and the rest which are spectators to the extremal transition. We then let $g_{I\bar{J}} = g_{I\bar{J}}(X^K)$ and $h_{\Lambda\Sigma} = h_{\Lambda\Sigma}(q^\Xi)$ be the special Kähler metric and quaternionic metric. Due to the gauging, a scalar potential is generated. The supergravity action is now regular in the neighborhood of singular loci since the inclusion of all light states repairs the IR divergences.

Attraction to conifold transition loci

Near the conifold locus, the scalar fields participating in the extremal transition are those in the vector multiplets of $U(1)^S$, X^i ($i = 1, \ldots, S$), and those in the R hypermultiplets charged under $U(1)^S$, $q^{\mathcal{A}u}$ ($\mathcal{A} = 1, \ldots, R;\ u = 1, 2, 3, 4$). The conifold locus can be represented by $X^i = 0 = q^{\mathcal{A}u}$ with suitable choice of parametrization. For simplicity we switch off the spectator scalar fields. In the neighborhood of the conifold locus, by performing power expansion in X^i and $q^{\mathcal{A}u}$ and imposing $U(1)^S$-isometry, we obtain the scalar part of the supergravity action to the lowest order [7]:

$$\begin{aligned} S =& \frac{1}{\kappa_{(4)}^2} \int d^4x \sqrt{-g} \Big[\frac{1}{2} R - g_{i\bar{j}} \partial X^i \partial \bar{X}^j - \nabla \mathcal{Q}^{\mathcal{A}\dagger} \nabla \mathcal{Q}^{\mathcal{A}} \\ & - g_c^2 \, e^{\mathcal{K}_V} \sum_{i,j} \Big(4 \sum_{\mathcal{A}} Q_i^{\mathcal{A}} Q_j^{\mathcal{A}} \, \bar{X}^i X^j \, \mathcal{Q}^{\mathcal{A}\dagger} \mathcal{Q}^{\mathcal{A}} + g^{i\bar{j}} \, \vec{D}_i \cdot \vec{D}_j \Big) \Big], \end{aligned} \tag{3.3}$$

where $Q_i^{\mathcal{A}}$ is the charge of the $\mathcal{A}$-th monopole under the i-th $U(1)$, g_c the gauge coupling constant. The Kähler potential $\mathcal{K}_V$, the special Kähler metric and the quaternionic metric in the above action are constant, taking their values at the conifold locus. Also we have defined the $SU(2)_{\mathcal{R}}$ doublet and the D-term:

$$\mathcal{Q}^{\mathcal{A}} = \begin{pmatrix} -q^{\mathcal{A}2} + i\, q^{\mathcal{A}1} \\ q^{\mathcal{A}3} + i\, q^{\mathcal{A}4} \end{pmatrix}, \quad \vec{D}_i = \sum_{\mathcal{A}} Q_i^{\mathcal{A}} \mathcal{Q}^{\mathcal{A}\dagger} \vec{\sigma} \mathcal{Q}^{\mathcal{A}}. \tag{3.4}$$

The action (3.3) describes an $\mathcal{N}_4 = 2$ supersymmetric Abelian gauge field theory formally coupled to gravity. We show that moduli are attracted to the conifold locus whether starting in the Coulomb branch or the Higgs branch.

• In the Coulomb branch, corresponding to the compactification on M, X^i ($i = 1, \ldots, S$) obtain nonzero VEV's, while $q^{\mathcal{A}u}$ ($\mathcal{A} = 1, \ldots, R;\ u = 1, 2, 3, 4$) have zero VEV. Thus the VEV's of X^i form S of the $h_{11}(M)$ Kähler moduli, parameterizing the Coulomb branch vacua together with the moduli fields which are spectators to the conifold transition. The free energy density is [7]

$$\mathcal{F} = -T^4 \Big[n_0 + \sum_s n_s G\Big(\frac{M_s}{T}\Big) \Big] + \mathcal{O}\big(e^{-\frac{M_{\min}}{T}}\big), \tag{3.5}$$

where n_0 and n_s count respectively the massless states and the light monopole states. Also $M_{\rm min}$ is the minimum mass of the states which never become massless in the neighborhood of the conifold locus. We let the temperature be much lower than this mass, $T \ll M_{\rm min}$, so that the contribution from massive states is exponentially supressed. In the argument of the G-function, M_s is the tree-level mass of the s-th light monopole state, which has the behavior $M_s \sim \mathcal{O}(X^i)$. Therefore at the conifold locus where $X^i = 0$, the free energy density reaches its local minimum. Thus the S Kähler moduli X^i are attracted to the conifold locus.

• In the Higgs branch, corresponding to the compactification on M', $q^{\mathcal{A}u}$ ($\mathcal{A} = 1, \ldots, R$; $u = 1, 2, 3, 4$) have nonzero VEV's subjected to the constraints $\vec{D}_i = 0$ modulo gauge orbits, so that they parameterize $R - S$ of the $h_{12}(M') + 1$ quaternionic directions in the complex structure moduli space. On the other hand, the VEV's of X^i vanish, and the vector multiplets containing X^i absorb S hypermultiplets to form S long massive vector multiplets. The free energy density takes the same form of Eq.(3.5), with $M_s \sim \mathcal{O}(q^{\mathcal{A}u})$. Thus the $4(R - S)$ hypermultiplet moduli $q^{\mathcal{A}u}$ are attracted to 0, corresponding to the conifold locus in the Higgs branch.

Attraction to non Abelian loci

Near the non Abelian locus the scalar fields relevant to the extremal transition are those in the $SU(N)$-vector multiplet, X^a ($a = 1, \ldots, N^2 - 1$), as well as those in the g hypermuliplets in the adjoint of $SU(N)$, $q^{a\mathcal{A}u}$ ($\mathcal{A} = 1, \ldots, g$; $u = 1, 2, 3, 4$), and we suppose $g > 1$.[1] The non Abelian loci can be parameterized as $X^a = 0 = q^{a\mathcal{A}u}$. Expanding in powers of X^a and $q^{a\mathcal{A}u}$, imposing $SU(N)$ isometry, we obtain the bosonic part of the supergravity action to the lowest order [7]:

$$\begin{aligned} S = \frac{1}{\kappa^2_{(4)}} \int d^4x \sqrt{-g} \Big[& \frac{1}{2} R - l^2 \nabla X^a \nabla \bar{X}^a - \nabla \mathscr{Q}^{a\dagger}_{\mathcal{A}} \nabla \mathscr{Q}^a_{\mathcal{A}} \\ & - g_c^2 \, e^{\mathcal{K}_V} \big\{ l^2 [X, \bar{X}]^2 + 4 [\bar{X}, q^{\mathcal{A}u}]^a [q^{\mathcal{A}u}, X]^a + l^{-2} \vec{D}^a \cdot \vec{D}^a \big\} \Big], \end{aligned} \tag{3.6}$$

where l is a nonzero constant. The $SU(2)_{\mathcal{R}}$ doublet $\mathscr{Q}^a_{\mathcal{A}}$ and the D-term $\vec{D}^a$ are defined as

$$\mathscr{Q}^a_{\mathcal{A}} = \begin{pmatrix} -q^{a\mathcal{A}2} + i\, q^{a\mathcal{A}1} \\ q^{a\mathcal{A}3} + i\, q^{a\mathcal{A}4} \end{pmatrix}, \quad \vec{D}^a = \sum_{\mathcal{A}, b, c} i\, f^{abc} \mathscr{Q}^{b\mathcal{A}\dagger} \vec{\sigma} \mathscr{Q}^{c\mathcal{A}}, \tag{3.7}$$

where f^{abc} are the structure constants of $SU(N)$. The action (3.6) thus describes an $\mathcal{N}_4 = 2$ $SU(N)$ super Yang-Mills field theory formally coupled to gravity. We show that moduli can be attracted to the non Abelian locus from either the Coulomb branch or the Higgs branch.

• In the Coulomb branch, corresponding to the compactification on M, all Cartan components $X^{\hat{a}}$ and $q^{\hat{a}\mathcal{A}u}$ ($\hat{a} = 1, \ldots, N - 1$; $\mathcal{A} = 1, \ldots, g$; $u = 1, 2, 3, 4$) acquire nonzero VEV's,

[1] When $g = 0$, the pure $SU(N)$ gauge theory has UV freedom, and is Abelian in the IR with the gauge group $U(1)^{N-1}$. This situation can be regarded as an example of the conifold case with $S = N - 1$ and $R = 0$. For $g = 1$, the $SU(N)$-vector multiplet and the only hypermultiplet in the adjoint representation combine into an $\mathcal{N} = 4$ gauge sector. This case is conformal and is already dealt with in Sec.2.

while all the non Cartan components vanish. Therefore $X^{\hat{a}}$ form $N-1$ of the $h_{11}(M)$ Kähler moduli, while $q^{\hat{a}\mathcal{A}u}$ form $4g(N-1)$ of the $4h_{12}(M)+4$ complex structure moduli. The free energy density takes the form of Eq.(3.5) with $M_s \sim \mathcal{O}\big(X^{\hat{a}}, q^{\hat{a}\mathcal{A}u}\big)$. Therefore the non Abelian locus where $X^{\hat{a}}$ and $q^{\hat{a}\mathcal{A}u}$ vanish is the local minimum of the free energy density. By consequence, $X^{\hat{a}}$ and $q^{\hat{a}\mathcal{A}u}$ are attracted to the non Abelian locus.

• In the Higgs branch, corresponding to the compactification on M'', $q^{a\mathcal{A}u}$ ($a=1,\dots,N^2-1$; $\mathcal{A}=1,\dots,g$; $u=1,2,3,4$) have nonzero VEV's satisfying the constraint $\vec{D}^a=0$ modulo gauge orbits, and they form $4(g-1)(N^2-1)$ of the $4h_{12}(M'')+4$ complex structure moduli. The scalars in the $SU(N)$-vector multiplet X^a vanish. The $SU(N)$-vector multiplet absorbs one hypermultiplet in the adjoint of $SU(N)$ and becomes a long massive vector multiplet. The free energy density is of the form Eq.(3.5), with $M_s \sim \mathcal{O}\big(q^{a\mathcal{A}u}\big)$. Thus the $(g-1)(N^2-1)$ complex structure moduli $q^{a\mathcal{A}u}$ are attracted to 0, corresponding to the non Abelian locus in the Higgs branch.

An example: stabilization at intersections of extremal transition loci

We analyze a 2-parameter example with heterotic dual, where the internal CY space is $M \in \mathbf{P}^4_{(1,1,2,2,6)}[12](2,128)$. Its mirror is defined by [19]

$$x_1^{12}+x_2^{12}+x_3^6+x_4^6+x_5^2-12\,\psi\,x_1x_2x_3x_4x_5-2\,\phi\,x_1^6x_2^6=0. \tag{3.8}$$

The complex coefficients ϕ and ψ are the two Kähler moduli (from the type IIA point of view). This model has at once a conifold locus with $R=S=1$, and an $SU(2)$-non Abelian locus with $g=2$. The latter leads to a Higgs branch corresponding to the CY space $M'' \in \mathbf{P}^5_{(1,1,1,1,1,3)}[2,6](1,129)$. These singular loci are defined by the vanishing of [19]

$$\Delta=\Delta_c\times\Delta_s=\big((1-z_1)^2-4z_1^2z_s\big)\times\big(1-4z_s\big), \tag{3.9}$$

where $\Delta_c=0$ defines the conifold locus, and $\Delta_s=0$ the non Abelian locus, with $z_1=-\frac{1}{864}\frac{\phi}{\psi^6}$, $z_s=\frac{1}{4\phi^2}$ a reparametrization of the Kähler moduli. The two singular loci intersect at two points: $\big(z_1,z_s\big)=\big(\frac{1}{2},\frac{1}{4}\big)$ and $\big(\infty,\frac{1}{4}\big)$, which are the favored points of moduli stabilization, since there is a maximal number of massless modes at these points. Thus sitting in the Coulomb branch, we can lift the whole Kähler moduli space and 2 of the $128+1$ quaternionic flat directions in the complex structure moduli space. Also in the Coulomb branch, the heterotic dual compactified on $K3\times T^2$ exists [20]. Therefore we can infer from the type II side the stabilization of the dual heterotic moduli. Especially since the whole vector multiplet moduli space is lifted, the heterotic dilaton, living in a vector multiplet, can be stabilized.

4 Summary and perspectives

We have studied moduli stabilization by thermal effects in the cosmological context. The breaking of supersymmetry generates a thermal free energy at one-loop level. The moduli are attracted to its local minima, where extra massless modes appear in the low energy spectrum. These extra massless states can either be perturbative or non perturbative. The scalar masses induced by such thermal effect are time-dependent, which ensures that the universe is radiation dominated at the exit of the intermediate era, so that the cosmological moduli problem does not arise.

Detailed analysis is carried out first to maximally supersymmetric heterotic strings in the weak coupling regime. It is reported for spacetime dimension $D \geq 4$ that all moduli except the dilaton are stabilized at enhanced gauge symmetry points, where the extra massless states are perturbative. Additionally for $D \geq 5$, the dilaton is frozen somewhere in the flat direction, while for $D = 4$, it has a logarithmic behavior. Passing to the dual type I picture using the S-duality, one finds that for $D = 4, 5$ ($D \geq 7$), all the internal type I moduli can be stabilized in the weak (strong) coupling regime, with the dilaton frozen somewhere in the flat direction. However for $D = 6$, where the S-duality map exchanges the heterotic (type I) dilaton with the type I (heterotic) internal volume, the internal volume is frozen in the flat direction and all other moduli including the dilaton are stabilized. The extra massless states are either non perturbative D-string states or perturbative open string states.

Another model studied is the type II strings compactified on CY three-folds. The moduli space admits particular loci where 2-cycles or 3-cycles in the internal CY manifold shrink to zero size, leading to conifold transition or non Abelian gauge symmetry. Extra massless $\mathcal{N}_4 = 2$ supermultiplets arise at these loci, inducing local minima to the one-loop free energy. The analysis is based on writing out the full effective action without integrating out the extra light states, so that the action is free of IR divergences. As a result, all type II moduli characterizing the deformation of the shrinking cycles are stabilized. More generally, the favored points in the moduli space are the intersection points of several such loci. An explicit example is given where moduli are stabilized at the intersection of a conifold transition locus and a non Abelian locus, where the entire Kähler moduli space is lifted. This implies in the dual heterotic picture that all vector multiplet moduli are stabilized, including the heterotic dilaton.

More work can be carried out on models with $\mathcal{N}_4 = 1$ supersymmetry, for instance, the type II models compactified on generalized CY spaces [21] including fluxes, branes and/or orientifold projections. As mentioned in the introduction, realistic models require a zero temperature spontaneous supersymmetry breaking mechanism, so that the $\mathcal{N}_4 = 1$ supersymmetry remains broken at low temperature. Thus it would be of interest to extend the orbifold model results in Refs. [9] to the context of generalized CY compactifications. Moreover, toroidal compactifications of type II strings in the presence of “gravito-magnetic” fluxes lead to thermal models free of Hagedorn-like divergences, and the induced cosmology has no initial singularity [13]. Therefore we can investigate the implementation of gravito-magnetic fluxes in the (generalized) CY compactifications, hopefully to obtain a theoretical framework giving a full account for both primordial and late-time universe.

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Moduli stabilization in type II Calabi-Yau compactifications at finite temperature

Lihui Liu and Hervé Partouche

Centre de Physique Théorique, Ecole Polytechnique,†
F–91128 Palaiseau cedex, France

lihui.liu@cpht.polytechnique.fr
herve.partouche@cpht.polytechnique.fr

Abstract

We consider the type II superstring compactified on Calabi-Yau threefolds, at finite temperature. The latter is implemented at the string level by a free action on the Euclidean time circle. We show that all Kähler and complex structure moduli involved in the gauge theories geometrically engineered in the vicinity of singular loci are lifted by the stringy thermal effective potential. The analysis is based on the effective gauged supergravity at low energy, without integrating out the non-perturbative BPS states becoming massless at the singular loci. The universal form of the action in the weak coupling regime and at low enough temperature is determined in two cases. Namely, the conifold locus, as well as a locus where the internal space develops a genus-g curve of A_{N-1} singularities, thus realizing an $SU(N)$ gauge theory coupled to g hypermultiplets in the adjoint. In general, we argue that the favored points of stabilization sit at the intersection of several such loci. As a result, the entire vector multiplet moduli space is expected to be lifted, together with hypermultiplet moduli. The scalars are dynamically stabilized during the cosmological evolution induced by the back-reaction of the thermal effective potential on the originally static background. When the universe expands and the temperature T drops, the scalars converge to minima, with damped oscillations. Moreover, they store an energy density that scales as T^4, which never dominates over radiation. The reason for this is that the mass they acquire at one-loop is of order the temperature scale, which is time-dependent rather than constant. As an example, we analyze the type IIA compactification on a hypersurface $\mathbb{P}^4_{(1,1,2,2,6)}[12]$, with Hodge numbers $h_{11} = 2$ and $h_{12} = 128$. In this case, both Kähler moduli are stabilized at a point, where the internal space develops a node and an enhanced $SU(2)$ gauge theory coupled to 2 adjoint hypermultiplets. This shows that in the dual thermal heterotic picture on $K3 \times T^2$, the torus modulus and the axio-dilaton are stabilized, though in a strong coupling heterotic regime.

† Unité mixte du CNRS et de l'Ecole Polytechnique, UMR 7644.

1 Introduction

The presence of moduli fields in supersymmetric compactifications of string theory leads to difficulties. Massless scalars are not only in contradiction with observations of the gravitational force (see for example [1]), they also lead to continuous parameters in the couplings and mass spectrum, implying a loss of predictability of the theory. Moreover, moduli spaces which are nothing but the flat directions of a scalar potential, often admit particular loci, where states generically massive become massless. In the literature, several mechanisms indicate these loci correspond to dynamically preferred values of the scalar vacuum expectations values (VEV's). In the context of M-theory or type II compactifications on Calabi-Yau (CY) spaces, it is shown in [2] that if the scalars are given initial conditions away from the minima of the potential, their temporal trajectories are attracted toward the loci of additional massless states. In [3], the scalar initial conditions are set along the flat directions, but with non-trivial velocities. The moduli motion induces particle productions, whose back-reaction implies again an attraction toward the same loci. However, if the scalars are initially along their flat directions and static, the above mechanisms are ineffective. Moreover, even in the cases they manage to dynamically select expectation values, the moduli fields remain massless at the end of the process, and additional massless scalars may even be present at these particular points.

On the contrary, the existence of flat directions in non-supersymetric theories is much more sparse [4, 5]. To avoid the presence of a very large cosmological constant, it is natural to focus on "no-scale models", which by definition are tree level backgrounds in Minkowski space, where supersymmetry is spontaneously broken [6]. If at the classical level the scale of supersymmetry breaking and other scalars are moduli fields, the associated flat directions are generically lifted at the quantum level, due to the generation of a non-trivial effective potential. In fact, any supersymmetric string compactification in flat space can lead to a no-scale model by switching on finite temperature. This can be done at the level of the conformal field theory on the worldsheet by compactifying the Euclidean time on a circle and modding out by the $\mathbb{Z}_2$ freely acting orbifold $(-1)^F\,\delta$, where F is the fermion number and δ is an order-two shift along the temporal circle. Physically, this is equivalent to imposing $(-1)^F$ boundary conditions along an Euclidean circle of perimeter equal to the inverse temperature [7,8]. In this case, the supersymmetry breaking scale is the temperature itself, while the effective potential is nothing but the free energy density $\mathcal{F}$.

The question of moduli stabilization in a universe filled with a gas of strings at thermal equilibrium is considered in [9, 10]. In [11], the case of the heterotic string compactified on a torus is analyzed at weak coupling. It is shown that at finite temperature, the points of enhanced gauge symmetry are minima of the free energy density, where all the internal moduli can be dynamically stabilized. There is no flat direction left (except for the dilaton) when the gauge group does not contain Abelian factors [4]. In the S-dual picture in type I, one finds that the light vector multiplets responsible for the enhancement of the gauge group are either perturbative or D-strings wrapped in the internal torus. In this case, the internal closed string moduli (Neuveu Schwarz-Neuveu Schwarz (NS-NS) and Ramond-Ramond (RR)) together with the open string Wilson lines are stabilized [11]. This indicates that BPS states becoming massless at some point in moduli space should be treated on equal footing, wether they are perturbative or not.

In the present work, we use this fact to lift flat directions in the case of type II compactifications on CY threefolds, when finite temperature is switched on. Compared to the heterotic or type I strings on tori, the number of conserved supercharges present at zero temperature is half and the moduli space in four dimensions is by far more complicated. It takes the form of a product $\mathcal{M}_V \times \mathcal{M}_H$ associated to Abelian vector multiplets and neutral hypermultiplets. Physically, these spaces realize the Coulomb and/or Higgs branches of Abelian and/or non-Abelian gauge theories [12–15]. Due to the fact that the type II dilaton sits in the universal hypermultiplet, the metric on $\mathcal{M}_H$ admits corrections in string coupling. On the contrary, the metric on $\mathcal{M}_V$ is exact at tree level, but is singular on loci where 2-cycles in type IIA (3-cycles in type IIB) vanish [16]. This fact is interpreted as the consequence of the existence of D2-branes (D3-branes) wrapping these cycles. They realize generically massive BPS states charged under the gauge group associated to the cycles and are integrated out, at the level of the low energy supergravity description. Therefore, when the cycles vanish and the BPS states become massless, the sigma-model metric on $\mathcal{M}_V$ develops a logarithmic divergence [12].[1]

In the present work, our aim is to argue that at finite temperature, the moduli adjust so that a maximum number of 2-cycles and 3-cycles vanish. To show this, we consider the low

[1]In the case of $\mathcal{N} = 4$, the moduli space describes Coulomb branches only, which are not corrected by the string coupling and do not present IR divergences, as follows from the vanishing of the gauge beta functions.

energy description of the models *without integrating out* the modes which become massless when the internal CY space is singular. Our approach can be summarized as follows. By convention, we present it in type IIA compactified on a CY M, rather than in the equivalent mirror picture in type IIB.

(i) In the vicinity of a singular point in $\mathcal{M}_V$, we identify the gauge group and charged matter arising from wrapped D2-branes on vanishing 2-cycles and include them in the tree level effective supergravity. The latter is insensitive to the temperature, since the Euclidean time circle can only be probed by loop corrections. The classical $\mathcal{N} = 2$ supergravity is based on a product of special Kähler and quaternionic manifolds $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$, whose metrics are unknown but satisfy constraints. First of all, they do not develop IR divergences and are thus regular. Second, they admit isometries we have to gauge in order to reproduce the gauge sector engineered geometrically.

(ii) The gauging introduces a scalar potential we determine explicitly in the neighborhood under consideration. Its flat directions admit Coulomb and often Higgs branches. Moving from the Coulomb phase to the Higgs phase corresponds to an extremal transition from the original internal space M to a topologically distinct CY space M', where vanishing 2-cycles have been deformed into 3-cycles.

(iii) In each branch, it is straightforward to determine from the potential the classical masses of the heavy states that belong to the gauge plus charged matter system. These masses depend on the moduli, which parameterize the flat directions associated to the Coulomb or Higgs phases.

(iv) In the weak coupling regime and at sufficiently low temperature, the above masses are the only things needed to compute the one-loop correction to the effective supergravity. The result amounts to adding the one-loop effective potential $\mathcal{F}$ to the classical action evaluated in some tree level vacuum. One finds that *all flat directions in the Coulomb and Higgs branches of the geometrically engineered gauge theory are lifted.*

(v) The one-loop action does not admit static solutions anymore. In other words, a cosmological evolution is induced by the thermal/quantum corrections. While the universe expands and the temperature drops, the moduli fields are attracted to the minimum of $\mathcal{F}$. The latter sits at the origin of the Coulomb and Higgs branches, where all tree level masses of the gauge plus matter system vanish. However, at one loop, all moduli masses are of order the temperature, while the gauge bosons remain massless. In fact, the only cosmological evolution with static moduli corresponds to the compactification on the singular configuration at the extremal transition between M and M'.

It should be stressed that since the moduli have "time-dependent masses" proportional to the temperature, they are never abundantly produced. Moreover, the energy stored in their oscillations around the minima of $\mathcal{F}$ is of order that of thermal radiation [11]. As a result, the cosmological moduli problem [17] is avoided. However to be realistic, the mechanism presented here has to be extended to models where $\mathcal{N} = 1$ supergravity is spontaneously broken at a scale M, and finite temperature T is switched on. This was done in [18] for orbifold compactifications, where it was found that the energy stored in the moduli oscillations is dominated by the contribution arising from radiation. In this case, the oscillations can be neglected and the moduli are stabilized at their minima. Moreover, when the temperature approaches the electroweak scale M_{ew} and the standard model Higgs mechanism is expected to take place, the evolution of the moduli masses should come to a halt. We expect these qualitative facts to remain valid in the more general class of interacting conformal field theories on the worldsheet.

In Section 2, we present in details the program (i)–(v) in the vicinity of a conifold locus. The gauge theory in this case is Abelian, with charged hypermultiplets [12]. We show that the Kähler moduli of M and complex structure moduli of M' involved in the extremal transition $M \leftrightarrow M'$ are attracted to this locus. A similar analysis is done in Section 3 in the neighborhood of points in the moduli space, where the internal CY M develops a genus-g curve of A_{N-1} singularities (with $g \geq 1$) and can be deformed into another CY space M'' (when $g \geq 2$). This system describes an $SU(N)$ gauge theory coupled to g hypermultiplets in the adjoint representation [13]. In this case, Kähler and complex structure moduli of M, together with complex structure moduli of M'', are attracted to the singular locus. In Section 4, we argue that for any given internal CY space, we expect our approach to apply to all Kähler moduli and most of the complex structure moduli[2]. However, the universal hypermultiplet scalars remain flat directions, at least in the weak coupling regime. To

[2]To be specific, it is not clear to us if a complex structure controlling the size of vanishing 3-cycles which *cannot* be blown up to 2-cycles can be stabilized.

illustrates our results, we consider the explicit example of a type IIA compactification on a CY M with Hodge numbers $(h_{11}, h_{12}) = (2, 128)$. The moduli space $\mathcal{M}_V$ admits two codimension one loci, where M develops either a node or a genus-2 curve of A_1 singularities. It follows that both Kähler moduli and some complex structure moduli can be stabilized at the intersection of these two loci. Given the fact that M is a $K3$-fibration [19], the dual heterotic description [20] on $K3 \times T^2$ at finite temperature is known. It follows that the T^2 modulus T_h and axio-dilaton S_h are stabilized in the strong coupling regime. Section 5 summarizes our results and presents our perspectives.

2 Stabilization at a conifold locus

In this Section, we consider the type II superstring compactified on M or M', two CY manifolds related by a conifold transition[3]. Our aim is to show that when finite temperature is switched on, the moduli involved in the extremal transition $M \leftrightarrow M'$ are lifted and attracted to the conifold locus, where they can be stabilized. We choose to present our analysis in type IIA. Due to mirror symmetry, the type IIB picture can be derived by exchanging the roles of 2-cycles with 3-cycles.

2.1 The geometrically engineered Abelian gauge theory

At zero temperature, the compactification on M yields an $\mathcal{N} = 2$ theory in four dimensions. The massless spectrum contains in the gravitational multiplet the metric $g_{\mu\nu}$ and the graviphoton A^0_μ (from the RR 1-form). When M is nonsingular, with Hodge numbers h_{11} and h_{12}, there are h_{11} Kähler deformations and $2h_{12}$ complex structure deformations of the CY metric. Combining these geometrical moduli with the reduction of the NS-NS 2-form and RR 3-forms leads to the bosonic content of h_{11} Abelian vector multiplets and h_{12} neutral hypermultiplets. Finally, the dilaton, the axion and the reduction of the 3-form on the unique $(3, 0)$ and $(0, 3)$ cycles realize the scalar content of the universal hypermultiplet. In total, the gauge group is $U(1)^{h_{11}+1}$. Moreover, due to the fact that $\mathcal{N} = 2$ supersymmetry forbids couplings between vector multiplets and neutral hypermultiplets, the moduli space is a Cartesian product $\mathcal{M}_V \times \mathcal{M}_H$. The vector multiplet moduli space $\mathcal{M}_V$ of complex di-

[3]In some particular case, there is no CY M' in which the conifold M can be deformed to.

mension h_{11} is a special Kähler manifold, which is exact at tree level since the dilaton sits in a hypermultiplet. On the contrary, the hypermultiplet moduli space $\mathcal{M}_H$ of real dimension $4(h_{12}+1)$ is a quaternionic manifold, which admits perturbative and non-perturbative corrections. Classically, it is a product manifold, where an $SU(2,1)/U(2)$ factor is associated to the universal hypermultiplet.

By definition, along a conifold locus of codimension S in $\mathcal{M}_V$, S homology classes of 2-cycles are vanishing, and $R \geq S$ representative 2-cycles in M are shrinking to isolated points called nodes [16]. The metric of $\mathcal{M}_V$ appears in the low energy effective σ-model description of the vector multiplets. To account for the fact that this metric is singular along the conifold locus and cannot be cured by quantum corrections in string coupling, a consistent picture has been proposed in Ref. [12]. In this work, it is supposed that generically massive states charged under the $U(1)^S$ gauge factors have been integrated out, and become massless along the conifold locus. Consistently, the σ-model metric develops an IR divergence[4]. Since the gauge bosons arise from the RR 3-form, the charged states must be D2-branes[5]. To be massless when the homology classes vanish, the D2-branes must be BPS and wrapped on the shrinking 2-cycles. To reproduce the precise coefficient of the logarithmic divergence, the charged states must be hypermultiplets. The wrapped D2-branes being point-like from a four-dimensional point of view, they are extremal black hole hypermultiplets.

Because the local neighborhood of a node looks like a cone whose base is $S^2 \times S^3$, the singular CY M is called a conifold. When $R > S$, this configuration can be a passage to another smooth CY M' obtained by deforming the shrinking 2-cycles into 3-cycles. This is the conifold transition, where the Hodge numbers h'_{11} and h'_{12} of M' satisfy [16]

$$h'_{11} = h_{11} - S \,, \qquad h'_{12} = h_{12} + R - S \,. \tag{2.1}$$

Denoting by $\mathcal{M}'_V \times \mathcal{M}'_H$ the moduli space of M', the extremal transition $M \leftrightarrow M'$ means $\mathcal{M}_V$ and $\mathcal{M}'_H$ are connected along the conifold locus. This geometrical picture matches perfectly the physical interpretation of the system in terms of a $U(1)^S$ gauge theory coupled to R hypermultiplets. $\mathcal{M}_V$ corresponds to the Coulomb branch: The scalars of the

[4]In gauge theory, this effect arises at one-loop in gauge coupling. Since the type II description of the $\mathcal{N}=2$ vector multiplet sector is exact in string coupling, the one-loop and non-perturbative corrections in gauge coupling are present at string tree level.

[5]An electric-magnetic duality can always be used to work with purely electric D2-branes, without introducing magnetic D4-branes.

Abelian vector multiplets have nontrivial VEV's, while the charged states arise from massive non-perturbative D2-branes. $\mathcal{M}'_H$ corresponds to the Higgs branch: The Abelian vector multiplets combine with S charged hypermultiplets to give S massive long vector multiplets[6]. The remaining $R-S$ charged hypermultiplets are massless perturbative states, which condense *i.e.* develop nontrivial VEV's along $\mathcal{M}'_H$.

Note that if the IR behavior of a $U(1)^S$ gauge theory is able to account for the singularity of the Kähler metric on $\mathcal{M}_V$, this does not mean the theory remains Abelian in the ultraviolet. Actually, type II compactifications on CY spaces with moduli sitting in the vicinity of a conifold locus can engineer geometrically $\mathcal{N}=2$ asymptotically free non-Abelian gauge theories [21, 22]. In this case, the non-Cartan gauge bosons are massive and our description of the theory in terms of an Abelian gauge group is valid for low enough energies or temperatures.

2.2 Tree level low energy description in gauged supergravity

To proceed, we determine the low energy description of the type IIA compactification on M (and eventually M' when a conifold transition is allowed), near a conifold configuration. At tree level in string coupling, the result is insensitive to temperature effects, since genus-zero worldsheets cannot probe an Euclidean time circle. To be consistent on both sides of the extremal transition, the $\mathcal{N}=2$ gauged supergravity we are looking for has to include all light and possibly massless degrees of freedom in the vicinity of the conifold locus, wether they are realized perturbatively or non-perturbatively from the string point of view.

It is convenient to start our discussion from the perspective of the type IIA compactification on M. The effective action is constructed in two steps. First, we consider the ungauged $\mathcal{N}=2$ supergravity coupled to h_{11} vector multiplets and $h_{12}+1+R$ hypermultiplets. The scalars of the vector multiplets live on a special Kähler manifold $\tilde{\mathcal{M}}_V$, while those of the hypermultiplets span a quaternionic manifold $\tilde{\mathcal{M}}_H$. Both metrics $g_{I\bar{J}}$ and $h_{\Lambda\Sigma}$ on $\tilde{\mathcal{M}}_V$ and $\tilde{\mathcal{M}}_H$ are unknown, but satisfy properties we are going to use. In particular, they are regular even when M is a conifold. By abuse of language, we will refer to the set of points in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$ corresponding to compactifications on conifold configurations of M as the

[6]It is not clear wether these massive multiplets are perturbative or non-perturbative.

conifold locus. There is a symplectic bundle over $\tilde{\mathcal{M}}_V$, whose holomorphic section admits electric and magnetic components we denote by $X^0, \ldots, X^{h_{11}}$ and $F_0, \ldots, F_{h_{11}}$. The former can be used as homogeneous coordinates on $\tilde{\mathcal{M}}_V$. Thus, at a given point $P_0 \in \tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$ along the conifold locus, at least one of them, say X^0, does not vanish and can be set to 1 in a whole neighborhood of P_0. The remaining complex components X^I ($I = 1, \ldots, h_{11}$) are then the vector multiplet scalars and special coordinates on $\tilde{\mathcal{M}}_V$. In the vicinity of P_0, we also denote by q^Λ ($\Lambda = 1, \ldots, 4(h_{12} + 1 + R)$) a system of real coordinates parameterizing the hypermultiplet scalar manifold $\tilde{\mathcal{M}}_H$.

In a second step, the charges of the hypermultiplets are introduced by gauging a $U(1)^S$ isometry group the quaternionic manifold $\tilde{\mathcal{M}}_H$ must satisfy. By convention, we label the vectors and scalar components of the gauged vector multiplets as A^i_μ and X^i ($i = 1, \ldots, S$), while the remaining X^p's ($p = S+1, \ldots, h_{11}$) denote the scalars of the ungauged ones. With these conventions, the tree level gauged supergravity action for the metric and scalars takes the following form [23],

$$S_{\text{tree}} = \int d^4x \sqrt{-g} \left\{ \frac{\mathcal{R}}{2} - g_{I\bar{J}}\, \partial_\mu X^I \partial^\mu \bar{X}^J - h_{\Lambda\Sigma} \nabla_\mu q^\Lambda \nabla^\mu q^\Sigma - \mathcal{V} \right\}, \tag{2.2}$$

where the covariant derivatives involve the non-trivial Killing vectors k^Λ_i,

$$\nabla_\mu q^\Lambda = \partial_\mu q^\Lambda + A^i_\mu k^\Lambda_i, \tag{2.3}$$

and the scalar potential $\mathcal{V}$ is given by

$$\mathcal{V} = 4h_{\Lambda\Sigma}\, k^\Lambda_i\, \bar{k}^\Sigma_j\, e^{\mathcal{K}} \bar{X}^i X^j + g^{I\bar{J}} f^i_I\, \bar{f}^j_J\, \mathcal{P}^x_i \mathcal{P}^x_j - 3\, e^{\mathcal{K}} \bar{X}^i X^j\, \mathcal{P}^x_i \mathcal{P}^x_j. \tag{2.4}$$

In this expression, $\mathcal{K}$ is the Kähler potential associated to the metric $g_{I\bar{J}} \equiv \partial_{X^I} \partial_{\bar{X}^J} \mathcal{K}$,

$$\mathcal{K} = -\ln \left[i \Big(F_0 - \bar{F}_0 + \bar{X}^I F_I - X^I \bar{F}_I \Big) \right], \tag{2.5}$$

and

$$f^i_I = \Big(\partial_{X^I} + \frac{1}{2} \partial_{X^I} \mathcal{K} \Big) \big(e^{\frac{1}{2}\mathcal{K}} X^i \big), \qquad \bar{f}^i_I = \Big(\partial_{\bar{X}^I} + \frac{1}{2} \partial_{\bar{X}^I} \mathcal{K} \Big) \big(e^{\frac{1}{2}\mathcal{K}} \bar{X}^i \big). \tag{2.6}$$

Moreover, for each Killing vector, there is an $SU(2)$-triplet of momentum maps $\mathcal{P}^x_i$, which are functions of q^Λ. They are related to the hyper-Kähler 2-forms K^x on $\tilde{\mathcal{M}}_H$ by the relation

$$2\, k^\Lambda_i K^x_{\Lambda\Sigma} = \nabla^{SU(2)}_\Sigma \mathcal{P}^x_i \equiv \partial_{q^\Sigma} \mathcal{P}^x_i + \epsilon^{xyz} \omega^y_\Sigma \mathcal{P}^z_i, \tag{2.7}$$

where ω^x is the connection of the $SU(2)$-bundle. The fact that the Killing vectors k_0^Λ and k_p^Λ vanish identically implies the associated momentum maps are covariantly constant and thus trivial, $\mathcal{P}_0^x \equiv \mathcal{P}_p^x \equiv 0$, as follows from the theorem recalled in Appendix C.

In a vacuum, the no-scale model has a vanishing potential, $\mathcal{V} = 0$. To identify the conifold locus on $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$, we use our knowledge of the geometrical realization of the gauge theory. When M is a conifold, all multiplets in the action (2.2) must be massless. For the q^Λ's to be massless, we see from the potential (2.4) that $\langle X^i \rangle = 0$ is required, while for the X^i's to be massless, the Killing vectors and momentum maps must have zeros, $\langle k_i^\Lambda \rangle = \langle \mathcal{P}_i^x \rangle = 0$. Thus, P_0 is fixed under the $U(1)^S$ isometries. In the remaining part of this Section, our aim is to expand the Lagrangian density in the action (2.2) around the point P_0.

Vector multiplets sector: We start with the vector multiplet sector and denote the coordinates of P_0 in $\tilde{\mathcal{M}}_V$ as $(X_0^i = 0, X_0^p)$. Smoothness of the σ-model Kähler metric in (2.2) allows us to write[7]

$$g_{I\bar{J}} = g_{I\bar{J}}^{(0)} + \mathcal{O}(X - X_0), \tag{2.8}$$

where $g_{I\bar{J}}^{(0)} \equiv g_{I\bar{J}}|_{P_0}$. Moreover, we will also need the finite value $\mathcal{K}^{(0)}$ of the Kähler potential (2.5) at P_0,

$$\mathcal{K}^{(0)} = -\ln\left[i\Big(F_0^{(0)} - \bar{F}_0^{(0)} + \bar{X}_0^p F_p^{(0)} - X_0^p \bar{F}_p^{(0)}\Big)\right], \tag{2.9}$$

in terms of which the f_I^i's defined in Eq. (2.6) can be expressed as,

$$f_I^i = e^{\frac{1}{2}\mathcal{K}^{(0)}}\, \delta_I^i + \mathcal{O}(X - X_0), \qquad \bar{f}_I^{\bar{i}} = e^{\frac{1}{2}\mathcal{K}^{(0)}}\, \delta_I^i + \mathcal{O}(X - X_0). \tag{2.10}$$

Hypermultiplets sector: The Taylor expansion in the hypermultiplet sector is more involved. In Appendix B, we show that on $\tilde{\mathcal{M}}_H$, there exists a new system of coordinates $c^{\mathcal{A}u}$ ($\mathcal{A} = 1, \ldots, R$; $u = 1, 2, 3, 4$), q^α ($\alpha = 4R + 1, \ldots, 4(h_{12} + 1 + R)$) such that P_0 is located at $(c^{\mathcal{A}u} = 0, q_0^\alpha)$ and the complex structures J^x, the quaternionic metric and the hyper-Kähler

[7] $\mathcal{O}(X - X_0)$ denotes without distinction holomorphic or antiholomorphic first order terms.

forms at P_0 are:

$$
\begin{aligned}
J^x\big|_{P_0} &= -\eta^{xu}{}_v \left(\frac{\partial}{\partial c^{\mathcal{A}u}} \otimes dc^{\mathcal{A}v}\right)\bigg|_{P_0} + \left(J^{x\alpha}{}_\beta \frac{\partial}{\partial q^\alpha} \otimes dq^\beta\right)\bigg|_{P_0}, \\
h\big|_{P_0} &= \frac{1}{2}\left(dc^{\mathcal{A}u} dc^{\mathcal{A}u}\right)\big|_{P_0} + \left(h_{\alpha\beta}\, dq^\alpha dq^\beta\right)\big|_{P_0}, \\
K^x\big|_{P_0} &= \frac{1}{4}\eta^x_{uv}\left(dc^{\mathcal{A}u} \wedge dc^{\mathcal{A}v}\right)\big|_{P_0} + \frac{1}{2}\left(K^x_{\alpha\beta}\, dq^\alpha \wedge dq^\beta\right)\big|_{P_0},
\end{aligned}
\tag{2.11}
$$

where $\eta^{xu}{}_v$ are 't Hooft symbols defined in Appendix A. In fact, the $(c^{\mathcal{A}1}, \ldots, c^{\mathcal{A}4})$'s are in the $\mathcal{A}^{\text{th}}$ hypermultiplets of charge $Q_i^{\mathcal{A}}$ under the i^{th} $U(1)$, while the remaining q^α's are the real components of the neutral ones[8]. In order to write the kinetic terms and scalar potential in the vicinity of P_0, we need the expansions of the metric, hyper-Kähler forms and Killing vectors[9],

$$
\begin{aligned}
&h_{\mathcal{A}u,\mathcal{B}v} = \frac{1}{2}\delta_{\mathcal{A}\mathcal{B}}\,\delta_{uv} + \mathcal{O}(q-q_0), \qquad h_{\mathcal{A}u,\alpha} = \mathcal{O}(q-q_0), \quad h_{\alpha\beta} = h^{(0)}_{\alpha\beta} + \mathcal{O}(q-q_0), \\
&K^x_{\mathcal{A}u,\mathcal{B}v} = \frac{1}{2}\delta_{\mathcal{A}\mathcal{B}}\,\eta^x_{uv} + \mathcal{O}(q-q_0), \qquad K^x_{\mathcal{A}u,\alpha} = \mathcal{O}(q-q_0), \quad K^x_{\alpha\beta} = K^{x(0)}_{\alpha\beta} + \mathcal{O}(q-q_0), \\
&k_i^{\mathcal{A}u} = Q_i^{\mathcal{A}u}\, t^u{}_v\, c^{\mathcal{A}u} + \mathcal{O}((q-q_0)^2) \;\; (\text{no sum over } \mathcal{A}), \qquad k_i^\alpha = \mathcal{O}((q-q_0)^2).
\end{aligned}
\tag{2.12}
$$

The first order terms of the Killing vectors involve $t^u{}_v$, a $U(1)$ generator acting on each hypermultiplet, and it is a matter of convention to choose $t^u{}_v \equiv -\bar{\eta}^{3u}{}_v$ (see Appendices A and B). The charges $Q_i^{\mathcal{A}}$ are determined by the underlying CY geometry. For this purpose, we define $(\alpha^0, \ldots, \alpha^{h_{11}})$ to be an homology basis of 2-cycles in M, among which α^i $(i = 1, \ldots, S)$ vanish at the conifold locus. We also denote by $\gamma^{\mathcal{A}}$ $(\mathcal{A} = 1, \ldots, R)$ the R 2-cycles which shrink to nodes and expand $\gamma^{\mathcal{A}} = n_i^{\mathcal{A}}\alpha^i$. Then, the computation of the effective action on the world volume of a D2-brane wrapped on $\gamma^{\mathcal{A}}$ shows that $Q_i^{\mathcal{A}} = n_i^{\mathcal{A}}$ [12].

To determine the momentum maps $\mathcal{P}_i^x$, we first use the facts that they vanish at P_0 and that the left hand side of Eq. (2.7) is first order to conclude that the $\mathcal{P}_i^x$'s are actually second order. Next, we write Eq. (2.7) as

$$
\frac{\partial \mathcal{P}_i^x}{\partial c^{\mathcal{A}u}} = Q_i^{\mathcal{A}}(\eta^x t)_{uv} c^{\mathcal{A}v} + \mathcal{O}((q-q_0)^2) \;\; (\text{no sum over } \mathcal{A}), \qquad \frac{\partial \mathcal{P}_i^x}{\partial q^\alpha} = \mathcal{O}((q-q_0)^2),
\tag{2.13}
$$

[8]To make contact with the notations introduced in Appendix B, we define $c^{\mathcal{A}u} = \sqrt{2}q^{\mathcal{A}u}$ in order for the charged hypermultiplets to have canonically normalized kinetic terms. Moreover, we keep arbitrary the basis vector $\partial/\partial q^\alpha$ in the sub-tangent plane at P_0 associated to the neutral hypermultiplets.

[9]$\mathcal{O}(q-q_0)$ denotes terms of order $c^{\mathcal{A}u}$ or $(q^\alpha - q_0^\alpha)$.

which we integrate to find

$$\mathcal{P}_i^x = -\frac{1}{2}\, Q_i^{\mathcal{A}}\, c^{\mathcal{A}u}(\eta^x t)_{uv} c^{\mathcal{A}v} + \mathcal{O}((q-q_0)^3). \tag{2.14}$$

Effective action: From Eqs (2.10), (2.12) and (2.14), we find that in the potential $\mathcal{V}$ in Eq. (2.4), the two first terms which are positive are of order four in $(X - X_0)$ or $(q - q_0)$, while the last one, which is negative and characteristic of supergravity, is of order six and thus negligible around the point P_0. To write the potential $\mathcal{V}$ in an explicitly $SU(2)_R$-invariant form, we introduce the doublets

$$\mathscr{C}^{\mathcal{A}} = \begin{pmatrix} i(c^{\mathcal{A}1} + ic^{\mathcal{A}2}) \\ (c^{\mathcal{A}3} + ic^{\mathcal{A}4})^* \end{pmatrix} \tag{2.15}$$

and obtain after some straightforward computation

$$\mathcal{V} = e^{\mathcal{K}^{(0)}} \Big(2\, Q_i^{\mathcal{A}} Q_j^{\mathcal{A}}\, \bar{X}^i X^j\, \mathscr{C}^{\mathcal{A}\dagger}\mathscr{C}^{\mathcal{A}} + \frac{1}{4}\, g^{(0)i\bar{\jmath}}\, D_i^x D_j^x\Big) + \cdots \quad \text{where} \quad D_i^x \equiv Q_i^{\mathcal{A}}\, \mathscr{C}^{\mathcal{A}\dagger}\sigma^x \mathscr{C}^{\mathcal{A}} \tag{2.16}$$

are $SU(2)_R$-triplets of D-terms, σ^x are the Pauli matrices and the ellipsis denote order five contributions in vector or hypermultiplet scalars.

In the end, close to a conifold configuration, the tree level effective action (2.2) associated to the type IIA superstring theory at finite temperature and compactified on either M or M' takes the final form,

$$\begin{aligned} S_{\text{tree}} = \int d^4x\, \sqrt{-g}\, \Big\{ & \frac{\mathcal{R}}{2} - g^{(0)}_{I\bar{J}}\, \partial_\mu X^I \partial^\mu \bar{X}^J - \frac{1}{2}\, \nabla_\mu c^{\mathcal{A}u} \nabla^\mu c^{\mathcal{A}u} - h^{(0)}_{\alpha\beta}\, \partial_\mu q^\alpha \partial^\mu q^\beta \\ & - e^{\mathcal{K}^{(0)}} \Big(2\, Q_i^{\mathcal{A}} Q_j^{\mathcal{A}}\, \bar{X}^i X^j\, \mathscr{C}^{\mathcal{A}\dagger}\mathscr{C}^{\mathcal{A}} + \frac{1}{4}\, g^{(0)i\bar{\jmath}}\, D_i^x D_j^x\Big) + \cdots \Big\}. \end{aligned} \tag{2.17}$$

It is interesting to note that the above action is that of the rigid $\mathcal{N} = 2$ supersymmetric Abelian gauge theory with charged hypermultiplets and formally coupled to gravity.

2.3 Lifting the Coulomb branch at one-loop

The tree level scalar potential (2.16) valid around P_0 admits flat directions. We recall that due to the no-scale structure of the theory, these directions are insensitive to the scale of spontaneous symmetry breaking, here identified with the temperature. However, the picture is drastically modified once quantum corrections are taken into account. In fact, a moduli and temperature dependent correction to the classically vanishing vacuum energy is generated and, as we are going to see, lifts all classical flat directions associated to the Abelian gauge theory. In the following, our analysis is restricted to a weak string coupling regime, with quantum corrections computed at one-loop.

In the neighborhood of the point P_0 with coordinates $(X_0^i = 0, X_0^p; c_0^{\mathcal{A}u} = 0, q_0^\alpha)$ in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$, the set of vacua of the action (2.17) is a Cartesian product between:

- The space parameterized by the "spectator moduli" X^p and q^α, which are only coupled gravitationally to the gauge theory. These scalars are coordinates along the conifold locus and reflect the arbitrariness in the choice of P_0 on it.
- The space of configurations of the X^i's and $c^{\mathcal{A}u}$'s, which are canceling the semi-definite positive potential $\mathcal{V}$. It is characterized by the constraints

$$\forall \mathcal{A} : \quad X^i Q_i^{\mathcal{A}} \mathscr{C}^{\mathcal{A}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ (no sum over } \mathcal{A}) \qquad \text{and} \qquad \forall x, i : \quad D_i^x = 0, \tag{2.18}$$

which admits Coulomb and Higgs branches.

The Coulomb branch of vacua corresponds to arbitrary values for the gauged vector multiplets scalars and vanishing VEV's for those in the charged hypermultiplets:

$$\text{Coulomb branch :} \qquad \Big\{ \big(X^i \text{ arbitrary}, c^{\mathcal{A}u} = 0 \big) \Big\} \times \Big\{ \big(X^p, q^\alpha \big) \text{ arbitrary} \Big\}. \tag{2.19}$$

To write the one-loop effective action at finite temperature in this branch, we evaluate the tree level part (2.17) in a background of the above form (2.19) and add the one-loop Coleman-Weinberg thermal effective potential $\mathcal{F}$,

$$S_{\text{1-loop}} = \int d^4x \sqrt{-g} \left\{ \frac{\mathcal{R}}{2} - g_{I\bar{J}}^{(0)} \partial_\mu X^I \partial^\mu \bar{X}^J - h_{\alpha\beta}^{(0)} \partial_\mu q^\alpha \partial^\mu q^\beta - \mathcal{F} + \cdots \right\}. \tag{2.20}$$

The computation of $\mathcal{F}$ is done in the Euclidean version of the theory, with time compactified on a circle of perimeter equal to the inverse temperature. All degrees of freedom are imposed $(-1)^F$ boundary conditions along the temporal circle, where F is the fermion number. For an arbitrary supersymmetric spectrum, the result is

$$\mathcal{F} = -\int_0^{+\infty} \frac{d\ell}{2\ell} \frac{1}{(2\pi\ell)^2} \sum_s e^{-\frac{M_s^2 \ell}{2}} \sum_{\tilde{m}_0} e^{-\frac{\tilde{m}_0^2}{2\ell T^2}} \big(1 - (-1)^{\tilde{m}_0} \big), \tag{2.21}$$

where T is the temperature and M_s is the classical mass of each degenerate pair s of boson and fermion. In this expression, ℓ is the proper time along the virtual loop wrapped $\tilde{m}_0$ times around the temporal circle and all dimension-full quantities are measured in Einstein frame. From a thermodynamical point of view, $\mathcal{F}$ is the free energy density associated to a perfect gas of bosons and fermions. In string compactifications where the supersymmetric spectrum at zero temperature is determined by a fully known conformal field theory, the expression (2.21) can be derived from a vacuum-to-vacuum string amplitude in Euclidean time (and suitable $(-1)^F$ boundary conditions). As an example, this is the case for the heterotic string on T^{10-D}, which leads to a D-dimensional model whose exact spectrum is known when $D \geq 6$ (so that no NS5-brane can wrap the internal torus) [11]. However in general, contributions of modes realized non-perturbatively from a string perspective cannot be captured by the CFT on the wordsheet. For instance, in the type I models S-dual to the above mentioned heterotic cases, the perturbative amplitude has contributions arising from fundamental open and closed strings and must be supplemented by additional terms associated to non-perturbative D1-branes running into the virtual loop. The role of BPS D1-branes wrapped on 1-cycles and becoming massless plays a role in stabilizing the type I moduli [11] similar to what we are going to find here for wrapped D2-branes.

Returning to our present case of interest, the light masses in the vicinity of P_0 along the Coulomb branch can be found by inspection of the bosonic action (2.17). The massless level includes the supergravity multiplet, the $I = 1, \dots, h_{11}$ vector multiplets and the $\alpha = 1, \dots, h_{12} + 1$ neutral hypermultiplets. Of course, this is not a surprise, since this is nothing but the perturbative massless spectrum arising from the type IIA compactification on a smooth CY manifold M. The light squared masses of the $\mathcal{A} = 1, \dots, R$ charged black hole hypermultiplets realized in the Coulomb phase as BPS D2-branes wrapped on vanishing 2-cycles are given by

$$M_{\mathcal{A}}^2 = 4\, e^{\mathcal{K}^{(0)}} |Q_i^{\mathcal{A}} X^i|^2 + \cdots \,, \tag{2.22}$$

where the dots stand for higher order terms in scalar fields. The leading term is consistent with the standard mass formula of BPS black holes [12, 24]. Close enough to P_0, all other masses M_s are bounded from below and heavier than the charged black holes: $M_s \geq M_{\min} > M_{\mathcal{A}}$. Table 1 summarizes the superfield content and associated scalar VEV's in the Coulomb branch[10].

	Scalars acquiring VEV's		Superfields				
	In vector multiplets	In hyper-multiplets	Vector multiplets			Hypermultiplets	
			Massless (moduli)	Massive short	Massive long	Massless (moduli)	Massive
Coulomb phase	X^i	none	S	0	0	0	R
Higgs phase	none	$\mathscr{C}^{\mathcal{A}}$ mod. gauge orbits such that $D_i^x = 0$	0	0	S	$R-S$	0

Table 1: Superfield contents in the Coulomb and Higgs branches (when $R > S$) associated to the $\mathcal{N} = 2\, U(1)^S$ gauge theory coupled to R hypermultiplets, which is encountered in the neighborhood of a conifold locus in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$. The scalars X^p and q^α of the massless spectator vector multiplets and hypermultiplets are not represented.

Integrating over ℓ, the free energy density (2.21) can be written as

$$\mathcal{F} = -T^4 \left\{ \Big(4 + 4h_{11} + 4(h_{12}+1)\Big) G(0) + 4 \sum_{\mathcal{A}} G\Big(\frac{M_{\mathcal{A}}}{T}\Big) + \mathcal{O}\big(e^{-\frac{M_{\rm min}}{T}}\big) \right\}, \tag{2.23}$$

where the function $G(x)$ is expressed in terms of a Bessel function of the second kind,

$$G(x) = 2 \sum_{k\in\mathbb{Z}} \Big(\frac{x}{2\pi|2k+1|}\Big)^2 K_2\big(x|2k+1|\big), \tag{2.24}$$

and $G(0)$ is Stefan's constant for radiation associated to a pair of massless boson and fermion,

$$G(0) = \frac{\Gamma(2)}{\pi^2} \sum_{k\in\mathbb{Z}} \frac{1}{|2k+1|^4} = \frac{\pi^2}{48}. \tag{2.25}$$

Moreover, the first factor 4 in Eq. (2.23) corresponds to the $2+2$ on shell degrees of freedom of the graviton and graviphoton, while the other factors 4 count the number of bosonic degrees of freedom in vector multiplets and hypermultiplets. In fact, for positive $x \geq 0$, the function $G(x)$ is maximum at the origin and decreases exponentially,

$$G(x) = G(0) - \frac{x^2}{16} + \mathcal{O}(x^4) \text{ when } x \ll 1, \quad G(x) \sim \Big(\frac{x}{2\pi}\Big)^{\frac{3}{2}} e^{-x} \text{ when } x \gg 1. \tag{2.26}$$

[10]See [25] for a general discussion about field contents following a Higgs mechanism in $\mathcal{N} = 2$ supersymmetric gauge theories.

As a result, all contributions $G(M_s/T)$ with $M_s \geq M_{\min}$ are exponentially suppressed, provided the temperature is low enough, $T < M_{\min}$, as indicated in Eq. (2.23).

Since the free energy density $\mathcal{F}$ depends on the black hole hypermultiplet masses given in Eq. (2.22), it acts as a non-trivial potential for the scalars X^i. The behavior of $G(x)$ at $x = 0$ implies $\mathcal{F}$ is minimum when all $M_{\mathcal{A}}$'s vanish *i.e.* $\forall \mathcal{A},\ Q_i^{\mathcal{A}} X^i = 0$. Due to the fact that the matrix $Q_i^{\mathcal{A}}$ is of rank S,[11] this can only happen at the conifold locus, $X^i = 0$. In other words, all classically flat directions X^i of the Coulomb branch are lifted, while the spectator scalars X^p and q^α parameterizing the conifold locus remain moduli. To find the one-loop masses $M_{i\text{1-loop}}$ of the fields X^i at their minimum, we consider the squared mass matrix

$$\Lambda^{\bar{I}}{}_{\bar{J}} = g^{(0)\bar{I}K} \left.\frac{\partial^2 \mathcal{F}}{\partial X^K \partial \bar{X}^J}\right|_{X^i=0} = \frac{T^2}{16}\, g^{(0)\bar{I}K}\, 4 \sum_{\mathcal{A}} \left.\frac{\partial^2 M_{\mathcal{A}}^2}{\partial X^K \partial \bar{X}^J}\right|_{X^i=0}, \tag{2.27}$$

which satisfies

$$\Lambda^{\bar{I}}{}_{\bar{\jmath}} = T^2 e^{\mathcal{K}^{(0)}} g^{(0)\bar{I}k}\, Q_k^{\mathcal{A}} Q_j^{\mathcal{A}}\,, \qquad \Lambda^{\bar{I}}{}_{\bar{p}} = 0. \tag{2.28}$$

Λ is diagonalizable, with S strictly positive eigenvalues $M^2_{i\text{1-loop}}$ and $h_{11} - S$ vanishing ones, so that the trace leads to

$$\sum_i M^2_{i\text{1-loop}} = T^2 e^{\mathcal{K}^{(0)}} g^{(0)\bar{\jmath}k}\, Q_k^{\mathcal{A}} Q_j^{\mathcal{A}}. \tag{2.29}$$

Thus, the scalars of the $U(1)^S$ vector multiplets acquire one-loop masses of order the temperature scale, while the gauge bosons remain massless and the full Abelian gauge theory $U(1)^{h_{11}+1}$ is unbroken. If the fact that vector multiplet scalars are getting masses is certainly a good thing, one may wonder if this result is not spoiled by the appearance of additional massless black hole hypermultiplets precisely at the minimum. In fact, the tree level masses of the scalars $c^{\mathcal{A}u}$ are vanishing at the conifold locus, but are acquiring non-trivial corrections of order T at the one-loop level, as will be seen in the next Section in the study of the Higgs branch.

For a homogeneous and isotropic universe, the one-loop energy density and pressure found by varying the action (2.20) are consistent with thermodynamics [26],

$$\rho = \mathcal{F} - T\,\frac{\partial \mathcal{F}}{\partial T}\,, \qquad P = -\mathcal{F}. \tag{2.30}$$

[11] Otherwise, some of the R vanishing 2-cycles would be linear combinations of the others and would not give independent degrees of freedom once wrapped with D2-branes.

When $T/M_{\rm min}$ is low enough, they satisfy at the conifold locus the state equation for radiation, $\rho = 3P$, as follows from Eq. (2.23). As a result, a particular solution to the equations of motion for the scale factor a, the temperature T and the scalars is

$$a(t) \propto \frac{1}{T(t)} \propto \sqrt{t}\ , \quad X^i \equiv c^{\mathcal{A}u} \equiv 0\ , \quad X^p, q^\alpha \text{ constant}, \tag{2.31}$$

where t is the cosmological time. This evolution describes an expanding universe filled with radiation and static scalars. Consistently, the temperature drops and guarantees $T \ll M_{\rm min}$ is satisfied at late times. More general solutions consistent with homogeneity and isotropy exist, as follows from the general analysis of Ref. [11]. They are characterized by damped oscillations of the fields X^i, which converge to their minimum at $X^i = 0$. The energy density stored in their oscillations and in the motion of the spectator moduli X^p and q^α scales as T^4. Thus, they do not dominate over radiation and the cosmological moduli problem [17] is avoided. This follows from the fact that the masses of the scalars X^i are actually proportional to the temperature and thus time-dependent [11]. To put a halt to the fall of these masses and obtain realistic models at low temperatures, one may follow the lines of Ref. [18]. At zero temperature, interesting models should be characterized by $\mathcal{N} = 1$ supersymmetry spontaneously broken at a scale M. Once finite temperature is taken into account, one finds the evolutions of the one-loop masses of the moduli, the temperature and the supersymmetry breaking scale M are attracted to a particular trajectory where they are proportional. When the temperature reaches the electroweak symmetry breaking scale M_{ew}, the moduli masses and M are expected to be stabilized around M_{ew}, while T keeps on decreasing.

2.4 Lifting the Higgs branch at one-loop

Our aim in this Section is to complete the analysis of the conifold locus by showing that the Higgs branch of the $U(1)^S$ gauge theory is lifted by the one-loop thermal effective potential. In this branch, the doublets $\mathscr{C}^\mathcal{A}$ are such that the D-terms in Eq. (2.18) vanish, while the $U(1)^S$ vector multiplet scalars have trivial VEV's,

$$\text{Higgs branch}: \quad \Big\{\big(X^i = 0, \mathscr{C}^\mathcal{A} \text{ such that } D_i^x = 0\big)\Big\} \times \Big\{\big(X^p, q^\alpha\big) \text{ arbitrary}\Big\}. \tag{2.32}$$

The $3S$ D-term constraints leave $4R - 3S$ flat directions among the charged scalars $c^{\mathcal{A}u}$'s, along which the $U(1)^S$ gauge group is Higgsed. S of the $4R-3S$ directions are orbits of the residual global $U(1)^S$ symmetry corresponding to physically equivalent vacua. We therefore introduce S gauge fixing conditions reflecting the fact that S would-be-Goldstone bosons are eaten by the massive vector fields. We are left with $4(R-S)$ flat directions, which can be arranged in $R-S$ massless neutral hypermultiplets. Clearly, for the Higgs branch to exist, $R>S$ is required. Moreover, the S Higgsed vector multiplets become massive and long by combining with the remaining S hypermultiplets. The superfield content and VEV's in the Higgs branch can be found in Table 1. Thus, besides the supergravity multiplet, the massless spectrum includes $h_{11}-S$ vector multiplets and $h_{12}+R-S+1$ neutral hypermultiplets, corresponding exactly to the type IIA compactification on the smooth CY manifold M', with Hodge numbers h'_{11} and h'_{12} given in Eq. (2.1).

To describe the one-loop effective action in the Higgs branch, it is convenient to parameterize the D-term flat directions with some coordinates ξ^m ($m=1,\dots,4(R-S)$) satisfying $Q_i^{\mathcal{A}}\,\mathscr{C}^{\mathcal{A}\dagger}(\xi)\,\sigma^x\,\mathscr{C}^{\mathcal{A}}(\xi)=0$ and such that the Jacobian matrix $\left(\frac{\partial c^{\mathcal{A}u}}{\partial\xi^m}\right)$ is of rank $4(R-S)$. We denote by ξ_0^m the origin of the Higgs branch *i.e.* the conifold locus. In a neighborhood of P_0, the one-loop effective action valid in the Higgs branch takes the form,

$$S_{\text{1-loop}}=\int d^4x\,\sqrt{-g}\left\{\frac{\mathcal{R}}{2}-g^{(0)}_{p\bar{q}}\,\partial X^p\partial\bar{X}^q-h^{(0)}_{mn}\,\partial\xi^m\partial\xi^n-h^{(0)}_{\alpha\beta}\,\partial q^\alpha\partial q^\beta-\mathcal{F}\right\},\qquad(2.33)$$

where we have defined

$$h^{(0)}_{mn}=\frac{1}{2}\,\frac{\partial c^{\mathcal{A}u}}{\partial\xi^m}\bigg|_{\xi_0}\,\frac{\partial c^{\mathcal{A}u}}{\partial\xi^n}\bigg|_{\xi_0},\qquad(2.34)$$

and the free energy density $\mathcal{F}$ is

$$\mathcal{F}=-T^4\left\{\Big(4+4h'_{11}+4(h'_{12}+1)\Big)G(0)+8\sum_i G\Big(\frac{M_i}{T}\Big)+\mathcal{O}\big(e^{-\frac{M_{\min}}{T}}\big)\right\}.\qquad(2.35)$$

The factor 8 in the above expression counts the number of boson/fermion pairs in the long vector multiplets of tree level mass M_i. The $\mathcal{O}\big(e^{-\frac{M_{\min}}{T}}\big)$ term includes all contributions of the modes whose masses cannot vanish in the neighborhood we are considering and thus admit a lower bound $M_{\min}>M_i$. For $T<M_{\min}$, these contributions are exponentially suppressed.

To proceed, we determine the sum of the squared masses of the long vector multiplets. This can be derived from the first term of the tree level potential $\mathcal{V}$ in Eq. (2.16), when the hypermultiplet scalars $c^{\mathcal{A}u}$ condense along the D-flat directions. At second order in scalar fields, the matrix of squared masses of the X^I's is

$$\Delta^{\bar{I}}{}_{\bar{\jmath}}=2e^{\mathcal{K}^{(0)}}g^{(0)\bar{I}k}\,Q_k^{\mathcal{A}}Q_j^{\mathcal{A}}\,c^{\mathcal{A}u}c^{\mathcal{A}u}+\cdots\,,\qquad\Delta^{\bar{I}}{}_{\bar{p}}=\cdots\,,\qquad(2.36)$$

whose trace gives

$$\sum_i M_i^2 = 2e^{\mathcal{K}^{(0)}} g^{(0)\bar{\jmath}k}\, Q_k^{\mathcal{A}} Q_j^{\mathcal{A}}\, c^{\mathcal{A}u} c^{\mathcal{A}u} + \cdots\,. \tag{2.37}$$

Alternatively, the second term of $\mathcal{V}$ in Eq. (2.16) can be used to compute the sum of the squared masses in the $c^{\mathcal{A}u}$-sector, when they condense. Consistently, the result in Eq. (2.37) is recovered.

In the vicinity of P_0, the thermal effective potential $\mathcal{F}$ is minimal when all masses M_i vanish. This occurs only at the origin of the Higgs branch, $c^{\mathcal{A}u} = 0$, *i.e.* along the conifold locus. Thus, all classically flat directions ξ^m are lifted, while the neutral components q^α remain moduli. The one-loop squared masses $M^2_{m\text{1-loop}} > 0$ of the fields ξ^m at their minimum $\xi = \xi_0$ are determined by the matrix

$$\begin{aligned}
\Lambda'^m{}_n &= \frac{1}{2} h^{(0)ml}\, \frac{\partial^2 \mathcal{F}}{\partial\xi^l \partial\xi^n}\bigg|_{\xi_0} = \frac{T^2}{16}\, \frac{1}{2} h^{(0)ml}\, 8 \sum_i \frac{\partial^2 M_i^2}{\partial\xi^l \partial\xi^n}\bigg|_{\xi_0} \\
&= T^2 e^{\mathcal{K}^{(0)}} g^{(0)\bar{\imath}j}\, Q_j^{\mathcal{A}} Q_i^{\mathcal{A}}\, h^{(0)ml}\, \frac{\partial c^{\mathcal{A}u}}{\partial \xi^l}\bigg|_{\xi_0} \frac{\partial c^{\mathcal{A}u}}{\partial \xi^n}\bigg|_{\xi_0}\,,
\end{aligned} \tag{2.38}$$

where we have used the fact that $c^{\mathcal{A}u}|_{\xi_0} = 0$ to obtain the last equality. The eigenvalues of Λ' are the desired squared masses we are looking for. Taking the trace, they satisfy

$$\sum_m M^2_{m\text{1-loop}} = T^2 e^{\mathcal{K}^{(0)}} g^{(0)\bar{\imath}j}\, Q_j^{\mathcal{A}} Q_i^{\mathcal{A}}\, h^{(0)nl}\, \frac{\partial c^{\mathcal{A}u}}{\partial \xi^l}\bigg|_{\xi_0} \frac{\partial c^{\mathcal{A}u}}{\partial \xi^n}\bigg|_{\xi_0}\,. \tag{2.39}$$

Thus, the ξ^m's have acquired a mass of order the temperature scale. Due to the arbitrariness in the choice of parametrization ξ^m of the D-flat directions, we conclude that all charged black hole hypermultiplets scalars $c^{\mathcal{A}u}$ have a mass of order T. Combining this fact with the result of Eq. (2.29), we see that at the one-loop level, all scalars involved in the $U(1)^S$ gauge theory coupled to R black hole hypermultiplets are acquiring masses at the conifold locus.

The particular homogeneous and isotropic evolution (2.31) found in the study of the Coulomb branch can now be seen as a limit case of another set of cosmological solutions, where the scalars ξ^m are oscillating with damping, as can be shown along the lines of Ref. [11]. The energy stored in the oscillations and in the motion of the spectator moduli q^α scales as T^4 and do not dominate over radiation.

3 Stabilization at a non-Abelian gauge symmetry locus

In this Section, our aim is to show how the moduli involved in extremal transitions realizing non-Abelian gauge theories can be stabilized, once finite temperature effects are taken into account. We specialize on the case of a geometric description of an $SU(N)$ gauge theory coupled to $g \geq 1$ hypermultiplets in the adjoint representation.

3.1 The geometrically engineered non-Abelian gauge theory

Our starting point is the type IIA theory compactified on a CY manifold M, which can develop a genus-g curve $\mathcal{C}$ of A_{N-1} singularities [13, 14]. Among the h_{11} homology 2-cycles, $N-1$ are realized by 2-spheres Γ_i ($i = 1, \dots, N-1$) in M, with intersection matrix corresponding to the Dynkin diagram of A_{N-1}, and with volume shrinking to zero when we sit on a codimension $N-1$ locus in the complexified Kähler moduli space $\mathcal{M}_V$. All connected 2-cycles built out of the Γ_i's are of the form $\Gamma_{ij} = \Gamma_i \cup \cdots \cup \Gamma_j$, for $1 \leq i \leq j \leq N-1$, and can be wrapped by BPS D2-branes or anti-D2-branes (obtained by reversing the orientations). The former (latter) are associated to the $(N^2 - N)/2$ positive (negative) roots of A_{N-1}, while the perturbative spectrum provides the remaining massless multiplets in the Cartan subalgebra. In the large volume limit of the curve $\mathcal{C}$, the model leads to an $\mathcal{N} = 2$ theory in six dimensions describing an $SU(N)$ gauge theory [27]. Thus, one can think about the four dimensional case as arising from an additional compactification on the curve $\mathcal{C}$ of genus g, which breaks further half of the supersymmetries. The result is an $\mathcal{N} = 2$ $SU(N)$ gauge theory coupled to g hypermultiplets in the adjoint representation [13].

In the following, we restrict our attention to the cases where $g \geq 1$. Actually, when $g = 0$, the pure $SU(N)$ gauge theory is asymptotically free and, as already mentioned at the end of Section 2.1, is Abelian in the IR, with gauge group $U(1)^{N-1}$. Thus, this situation is nothing but a particular example of the conifold case we have already studied, for $R = S = N-1$. For $g = 1$, the vector and hypermultiplet in the adjoint representation combine into an $\mathcal{N} = 4$ $SU(N)$ gauge sector. This case is conformal and has already been considered in Ref. [11], yielding to an attraction of the moduli at the origin of the Coulomb branches, thus restoring the full non-Abelian symmetry. On the contrary, new physics is encountered for $g \geq 2$, since the $SU(N)$ gauge theory is non-asymptotically free and moreover admits

Coulomb and Higgs branches.

The above phases are realized geometrically by compactifying on the original manifold M or on a distinct CY M'' related to M by extremal transition, $M \leftrightarrow M''$. It is instructive to recover the Hodge numbers h''_{11} and h''_{12} of M'' derived by deforming the vanishing 2-cycles into finite volume 3-cycles [13] from the gauge theory point of view. On a smooth CY M, which corresponds to a generic point in the Coulomb branch[12], the $SU(N)$ gauge group is spontaneously broken to $U(1)^{N-1}$, beside a remaining "spectator" $U(1)^n$ factor, where $n = h_{11} - (N-1) + 1$. The $h_{12}+1$ massless hypermultiplets include the $g(N-1)$ perturbative Cartan components of the g adjoint matter representations, together with m "spectator" hypermultiplets, $m = h_{12} - g(N-1) + 1$. The left-over $N^2 - N$ non-Cartan vector and $g(N^2-N)$ matter multiplets are massive. On the other hand, for the compactification on M'' to reproduce the spectrum in the Higgs branch, one must have $h''_{11}+1 = n$ Abelian vector multiplets, and $h''_{12}+1 = (g-1)(N^2-1)+m$ massless hypermultiplets. The remaining N^2-1 matter multiplets combine with the $SU(N)$ vector multiplets into long massive vector multiplets. As a result, one obtains [13]

$$h''_{11} = h_{11} - (N-1)\ , \qquad h''_{12} = h_{12} + (g-1)(N^2-1) - g(N-1)\,. \tag{3.1}$$

3.2 Tree level low energy description in gauged supergravity

At finite temperature, the one-loop low energy effective action of the type IIA theory compactified on either M or M'' can be decomposed in two parts. The classical one, which is independent of T, and the one-loop Coleman-Weinberg thermal effective potential evaluated in some classical background. In this Section, we focus on the tree level action, which can be described by an $\mathcal{N}=2$ gauged supergravity. The latter, to be valid in the neighborhood of the extremal transition $M \leftrightarrow M''$, has to include explicitly the whole set of light degrees of freedom, including the $SU(N)$ gauge sector coupled to g hypermultiplets in the adjoint.

Due to the relation (3.1), we can organize our discussion in terms of the Hodge numbers of M. The ungauged supergravity we start with contains $h_{11}+N^2-N$ vector multiplets and

[12]When some non-trivial VEV's in the Cartan subalgebra coincide, $SU(N)$ is broken to a product of non-Abelian SU groups and $U(1)$ factors, with total rank equal to $N-1$. Geometrically, this corresponds to singular configurations of M, where some (but not all) of the $N-1$ 2-spheres are vanishing. These circumstances are encountered in loci in the moduli space that admit the full enhanced $SU(N)$ enhanced symmetry locus as a submanifold.

$h_{12}+1+g(N^2-N)$ hypermultiplets, in order to take into account the non-Cartan generators, which arise from solitonic D2-branes when the compactification is on M. The scalar fields span a product manifold $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$, with special Kähler and quaternionic metrics $g_{I\bar{J}}$ and $h_{\Lambda\Sigma}$. As in the case of the conifold locus, we refer to the set of points in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$ associated to compactifications developing the curve of A_{N-1} singularities as the non-Abelian locus. In the symplectic bundle $(X^0, X^I; F_0, F_I)$ $(I = 1, \ldots, h_{11} + N^2 - N)$, the electric components are homogeneous coordinates of $\tilde{\mathcal{M}}_V$. Therefore, in the neighborhood of a given point P_0 on the non-Abelian locus, we can set one of the X-entries to 1, say X^0, and work with special coordinates X^I. Furthermore, we choose a chart q^Λ $\big(\Lambda = 1, \ldots, 4(h_{12} + 1 + g(N^2 - N))\big)$ of real coordinates on the hypermultiplet manifold $\tilde{\mathcal{M}}_H$, whose properties will be specified shortly.

By construction, the metrics $g_{I\bar{J}}$ and $h_{\Lambda\Sigma}$ are non-singular and admit a subgroup $SU(N)$ of isometries we now gauge. We choose the vectors and scalars of the gauged vector multiplets to be labeled as A^a_μ and X^a $(a = 1, \ldots, N^2 - 1)$ and denote the left-over "spectator" scalars as X^p $(p = N^2, \ldots, h_{11}+N^2-N)$. Restricting for simplicity to the metric and scalar degrees of freedom, the $\mathcal{N} = 2$ tree level gauged supergravity action is

$$S_{\text{tree}} = \int d^4x\, \sqrt{-g}\left\{\frac{\mathcal{R}}{2} - g_{I\bar{J}} \nabla_\mu X^I \nabla^\mu \bar{X}^J - h_{\Lambda\Sigma} \nabla_\mu q^\Lambda \nabla^\mu q^\Sigma - \mathcal{V}\right\}, \tag{3.2}$$

where the covariant derivatives are expressed in terms of the non-trivial Killing vectors k^I_a and k^Λ_a acting on $\tilde{\mathcal{M}}_V$ and $\tilde{\mathcal{M}}_H$,

$$\nabla_\mu X^I = \partial_\mu X^I + A^a_\mu k^I_a\,, \qquad \nabla_\mu \bar{X}^I = \partial_\mu \bar{X}^I + A^a_\mu \bar{k}^I_a\,, \qquad \nabla_\mu q^\Lambda = \partial_\mu q^\Lambda + A^a_\mu k^\Lambda_a. \tag{3.3}$$

The scalar potential $\mathcal{V}$ takes the form [23]

$$\mathcal{V} = \big(g_{I\bar{J}}\, k^I_a \bar{k}^J_b + 4 h_{\Lambda\Sigma}\, k^\Lambda_a\, k^\Sigma_b\big)\, e^{\mathcal{K}} \bar{X}^a X^b + g^{I\bar{J}} f^a_I\, \bar{f}^b_J\, \mathcal{P}^x_a \mathcal{P}^x_b - 3\, e^{\mathcal{K}} \bar{X}^a X^b\, \mathcal{P}^x_a \mathcal{P}^x_b\,, \tag{3.4}$$

where the Kähler potential $\mathcal{K}$ is defined as in Eq. (2.5) and

$$f^a_I = \Big(\partial_{X^I} + \frac{1}{2}\partial_{X^I}\mathcal{K}\Big)\big(e^{\frac{1}{2}\mathcal{K}} X^a\big), \qquad \bar{f}^a_I = \Big(\partial_{\bar{X}^I} + \frac{1}{2}\partial_{\bar{X}^I}\mathcal{K}\Big)\big(e^{\frac{1}{2}\mathcal{K}} \bar{X}^a\big). \tag{3.5}$$

The triplets of momentum maps appearing in Eq. (3.4) are associated to the non-trivial Killing vectors acting on the quaternionic manifold $\tilde{\mathcal{M}}_H$, and are related to the hyper-Kähler 2-forms K^x,

$$2\, k^\Lambda_a K^x_{\Lambda\Sigma} = \nabla^{SU(2)}_\Sigma \mathcal{P}^x_a \equiv \partial_{q^\Sigma} \mathcal{P}^x_a + \epsilon^{xyz} \omega^y_\Sigma \mathcal{P}^z_a. \tag{3.6}$$

Utilizing Appendix C, the momentum maps $\mathcal{P}_0^x \equiv \mathcal{P}_p^x$ vanish identically, since the associated Killing vectors k_0^Λ and k_p^Λ are trivial.

The vacua satisfy $\mathcal{V} = 0$, and in particular the non-Abelian locus in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$, where all vector multiplets and hypermultiplets we have introduced are massless. This is the case for the q^Λ's if $\langle X^a \rangle = 0$, and for the X^a's if the Killing vectors and momentum maps vanish, $\langle k_a^\Lambda \rangle = \langle \mathcal{P}_a^x \rangle = 0$. Thus, the point P_0 and actually the non-Abelian locus on which it sits are fixed by the isometries. In the following, our aim is to derive the Taylor expansion of the Lagrangian density of the action (3.2) at P_0.

Vector multiplets sector: Let us denote the coordinates of P_0 in $\tilde{\mathcal{M}}_V$ as $(X_0^a = 0, X_0^p)$ and begin our discussion with the vector multiplet sector. The infinitesimal isometry action on the scalars, $\delta X^I = \epsilon^a k_a^I$, is generated by Killing vectors satisfying the $su(N)$ algebra,

$$\left[k_a^I \partial_{X^I} \;,\; k_b^J \partial_{X^J}\right] = f^{abc}\, k_c^I \partial_{X^I}, \tag{3.7}$$

where f^{abc} are structure constants. Since the X^a's are in the adjoint representation and the Killing vectors are vanishing at P_0, we conclude that at lowest order,

$$k_a^b = f^{abc} X^c + \mathcal{O}((X - X_0)^2), \qquad k_a^p = \mathcal{O}((X - X_0)^2). \tag{3.8}$$

Utilizing Killing's equation, the above expressions can be used to constraint the zero$^{\text{th}}$ order of the Kähler metric,

$$\forall I, \bar{J}, a: \quad g_{I\bar{K}}\, D_{\bar{J}} \bar{k}_a^K + g_{K\bar{J}}\, D_I k_a^K = 0 \quad \Longrightarrow \quad g_{I\bar{b}}^{(0)}\, \frac{\partial(f^{abc}\bar{X}^c)}{\partial \bar{X}^J} + g_{b\bar{J}}^{(0)}\, \frac{\partial(f^{abc}X^c)}{\partial X^I} = 0, \tag{3.9}$$

where D is the covariant derivative on the complex manifold $\tilde{\mathcal{M}}_V$. Taking $(I, \bar{J}) = (d, \bar{e})$ leads to $[g^{(0)}, T^a] = 0$, where $(T^a)_{bc} = -if^{abc}$ are the $SU(N)$ generators. Therefore, $g_{a\bar{b}}^{(0)}$ is proportional to the identity matrix. On the other hand, the choice $(I, \bar{J}) = (d, \bar{p})$ yields $g_{e\bar{p}}^{(0)} = 0$. Altogether, we conclude that there exists a constant $l^2 > 0$ such that

$$g_{a\bar{b}} = l^2 \delta_{a\bar{b}} + \mathcal{O}(X - X_0), \qquad g_{a\bar{p}} = \mathcal{O}(X - X_0), \qquad g_{p\bar{q}} = g_{p\bar{q}}^{(0)} + \mathcal{O}(X - X_0). \tag{3.10}$$

Finally, the Kähler potential at P_0 takes the form given in Eq. (2.9), in term of which we have

$$f_I^a = e^{\frac{1}{2}\mathcal{K}^{(0)}}\, \delta_I^a + \mathcal{O}(X - X_0), \qquad \bar{f}_I^a = e^{\frac{1}{2}\mathcal{K}^{(0)}}\, \delta_I^a + \mathcal{O}(X - X_0). \tag{3.11}$$

Hypermultiplets sector: The matter sector necessitates more technical manipulations. Again, we start with the Killing vectors, whose action on $\tilde{\mathcal{M}}_H$ must satisfy

$$\left[k_a^\Lambda \partial_{q^\Lambda}\ ,\ k_b^\Sigma \partial_{q^\Sigma}\right] = f^{abc}\, k_c^\Lambda \partial_{q^\Lambda}. \tag{3.12}$$

Since the geometry of the compactification is telling us that this algebra is realized by $4g$ adjoint representations, we know we can single out a coordinate system $(q^{a\lambda}, q^\alpha)$, where $a = 1, \ldots, N^2-1$ and $\lambda = 1, \ldots, 4g$, and $\alpha = 4g(N^2-1)+1, \ldots, 4(h_{12}+1+g(N^2-N))$. Physically, the $4g$ adjoint representations are the components of the g hypermultiplets that are transforming under $SU(N)$, while the remaining q^α's are singlets of $SU(N)$ and therefore referred to as the components of the "spectator" hypermultiplets. Denoting the coordinates of P_0 in $\tilde{\mathcal{M}}_H$ as $(q_0^{a\lambda}, q_0^\alpha)$, where the Killing vectors are vanishing, the latter can be written at lowest order as

$$k_a^{b\lambda} = f^{abc}(q^{c\lambda} - q_0^{c\lambda}) + \mathcal{O}((q-q_0)^2), \qquad k_a^\alpha = \mathcal{O}((q-q_0)^2). \tag{3.13}$$

To write down the form of the quaternionic metric at P_0, we make use of Killing's equation, where we denote interchangeably the coordinate system as q^Λ or $(q^{a\lambda}, q^\alpha)$,

$$\forall \Lambda, \Sigma, a: \quad h_{\Lambda\Xi}\, D_\Sigma k_a^\Xi + h_{\Sigma\Xi}\, D_\Lambda k_a^\Xi = 0 \quad \Longrightarrow \quad h^{(0)}_{\Lambda, b\lambda} \frac{\partial(f^{abc} q^{c\lambda})}{\partial q^\Sigma} + h^{(0)}_{\Sigma, b\lambda} \frac{\partial(f^{abc} q^{c\lambda})}{\partial q^\Lambda} = 0. \tag{3.14}$$

In the above equation, D is now the covariant derivative on $\tilde{\mathcal{M}}_H$. The choice $(\Lambda, \Sigma) = (d\rho, e\sigma)$ gives $[h^{(\rho\sigma)}, T^a] = 0$, where we have defined $h^{(\rho\sigma)}_{db} \equiv h^{(0)}_{d\rho, b\sigma}$, which is therefore proportional to the identity matrix. Moreover, for $(\Lambda, \Sigma) = (d\rho, \alpha)$ one obtains $h^{(0)}_{\alpha, e\rho} = 0$. In total, we have at this stage

$$h_{a\lambda, b\rho} = \delta_{ab} h_{\lambda\rho} + \mathcal{O}(q-q_0), \qquad h_{a\lambda, \alpha} = \mathcal{O}(q-q_0), \qquad h_{\alpha\beta} = h^{(0)}_{\alpha\beta} + \mathcal{O}(q-q_0), \tag{3.15}$$

where $h_{\lambda\rho}$ is a constant metric.

To characterize the hyper-Kähler forms K^x at P_0, we use the fact the quaternionic structure must be preserved by the isometric flow, up to $SU(2)$ transformations. This is formulated by saying that there exist sections W_a^x of the $SU(2)$-bundle over $\tilde{\mathcal{M}}_H$, such that

$$\mathcal{L}_a K^x = \epsilon^{xyz} K^y W_a^z, \tag{3.16}$$

where $\mathcal{L}_a$ are the Lie derivatives with respect to the Killing vectors $k_a^\Lambda \partial_{q^\Lambda}$. The W_a^x's are called compensators and we first determine their value at P_0. To do so, we consider the relation [23]

$$\mathcal{L}_a \omega^x = \nabla^{SU(2)} W_a^x \equiv dW_a^x + \epsilon^{xyz} \omega^y W_a^z. \tag{3.17}$$

Starting from the definition $\mathcal{L}_a \omega^x = d\, i_a \omega^x + i_a d\omega^x$, where $d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z \equiv \Omega^x = \lambda K^x$ is the $SU(2)$ curvature proportional to the hyper-Kähler forms, and using $i_a(\omega^y \wedge \omega^z) = i_a \omega^y\, \omega^z - \omega^y\, i_a \omega^z$, we obtain from Eq. (3.17)

$$\nabla^{SU(2)} W_a^x = \lambda i_a K^x + \nabla^{SU(2)} i_a \omega^x. \tag{3.18}$$

Combining with Eq. (3.6), we find

$$\nabla^{SU(2)} \left(\frac{\lambda}{2} \mathcal{P}_a^x + W_a^x - i_a \omega^x \right) = 0 \qquad \Longrightarrow \qquad \frac{\lambda}{2} \mathcal{P}_a^x + W_a^x - i_a \omega^x = 0, \tag{3.19}$$

by virtue of Theorem 3 in Appendix C. Since both $\mathcal{P}_a^x$ and $k_a^\Lambda \partial_{q^\Lambda}$ vanish at P_0, we conclude that $W_a^x(q_0) = 0$ as well. We are now in a position to use efficiently Eq. (3.16) we rewrite in components as,

$$\forall \Lambda, \Sigma, a: \quad k_a^\Xi\, \partial_{q^\Xi} K^x_{\Lambda\Sigma} + (\partial_{q^\Lambda} k_a^\Xi) K^x_{\Xi\Sigma} + (\partial_{q^\Sigma} k_a^\Xi) K^x_{\Xi\Lambda} = \epsilon^{xyz} K^y_{\Sigma\Lambda} W_a^z. \tag{3.20}$$

At $q = q_0$, this relation with $(\Lambda, \Sigma) = (b\lambda, c\rho)$ gives $[K^{x(\lambda\rho)}, T^a] = 0$, where we have defined $K^{x(\lambda\rho)}_{bc} \equiv K^{x(0)}_{b\lambda, c\rho}$,which is thus proportional to the identity matrix. With $(\Lambda, \Sigma) = (b\lambda, \alpha)$, one obtains instead $K^{x(0)}_{c\lambda,\alpha} = 0$, so that

$$K^x_{a\lambda, b\rho} = \delta_{ab} K^x_{\lambda\rho} + \mathcal{O}(q - q_0), \qquad K^x_{a\lambda,\alpha} = \mathcal{O}(q - q_0), \qquad K^x_{\alpha\beta} = K^{x(0)}_{\alpha\beta} + \mathcal{O}(q - q_0), \tag{3.21}$$

where $K^x_{\lambda\rho}$ is an antisymmetric constant matrix.

The triplet of complex structures are related to the hyper-Kähler forms by the definition (B.4), so that $J^{x\Lambda}{}_\Sigma = -h^{\Lambda\Xi} K^x_{\Xi\Sigma}$. Using Eqs (3.15) and (3.21), we obtain in the vicinity of P_0,

$$J^{xa\lambda}{}_{b\rho} = \delta^a_b J^{x\lambda}{}_\rho + \mathcal{O}(q - q_0), \qquad J^{xa\lambda}{}_\alpha = \mathcal{O}(q - q_0), \qquad J^{x\alpha}{}_\beta = J^{x(0)\alpha}{}_\beta + \mathcal{O}(q - q_0), \tag{3.22}$$

where $J^{x\lambda}{}_\rho = -h^{\lambda\sigma} K^x_{\sigma\rho}$. Using the fact that the $J^{x\Lambda}{}_\Sigma$'s are satisfying the quaternionic algebra (Eq. (B.1) with $n = h_{12} + 1 + g(N^2 - N)$) everywhere on $\tilde{\mathcal{M}}_H$ and thus at P_0, we find ${J^{x\lambda}}_\sigma$ are also triplets of complex structures (Eq. (B.1) with $n = g$) in each of the N^2-1 sub-tangent planes $\mathcal{T}_{0a}$ at P_0 spanned by $\partial/\partial_{q^{a\lambda}}$ (at fixed a), endowed with the metric $h_{\lambda\rho}$. Applying Theorem 1 (see Appendix B) in $\mathcal{T}_{0a}$, there exists a new basis $e_{a\mathcal{A}u}$ in $\mathcal{T}_{0a}$ and its dual basis $\theta^{a\mathcal{A}u}$ in $\mathcal{T}^*_{0a}$, where $\mathcal{A} = 1, \dots, g$ and $u = 1, 2, 3, 4$, such that ${J^{x\lambda}}_\rho$, $h_{\lambda\rho}$ and $K^x_{\lambda\rho}$ take canonical forms. This local basis defines a new set of coordinates $c^{a\mathcal{A}u}$ such that $dc^{a\mathcal{A}u}\big|_{P_0} = \sqrt{2}\,\theta^{a\mathcal{A}u}$ and $c^{a\mathcal{A}u}\big|_{P_0} = 0$, which greatly simplifies the form of the metric, hyper-Kähler forms and Killing vectors on $\tilde{\mathcal{M}}_H$. Using Eqs (3.15), (3.21) and (3.13), we obtain in the neighborhood of P_0,[13]

$$\begin{aligned}
&h_{a\mathcal{A}u,b\mathcal{B}v} = \frac{1}{2}\delta_{ab}\delta_{\mathcal{A}\mathcal{B}}\,\delta_{uv} + \mathcal{O}(q-q_0), \quad h_{a\mathcal{A}u,\alpha} = \mathcal{O}(q-q_0), \quad h_{\alpha\beta} = h^{(0)}_{\alpha\beta} + \mathcal{O}(q-q_0),\\
&K^x_{a\mathcal{A}u,b\mathcal{B}v} = \frac{1}{2}\delta_{ab}\delta_{\mathcal{A}\mathcal{B}}\,\eta^x_{uv} + \mathcal{O}(q-q_0), \quad K^x_{a\mathcal{A}u,\alpha} = \mathcal{O}(q-q_0), \quad K^x_{\alpha\beta} = K^{x(0)}_{\alpha\beta} + \mathcal{O}(q-q_0),\\
&k_a^{b\mathcal{A}u} = f^{abc}\,c^{c\mathcal{A}u} + \mathcal{O}((q-q_0)^2), \qquad k_a^\alpha = \mathcal{O}((q-q_0)^2).
\end{aligned} \tag{3.23}$$

Finally, we know the momentum maps $\mathcal{P}^x_a$ are vanishing at P_0. Moreover, since the left hand side of Eq. (3.6) is first order, the $\mathcal{P}^x_a$'s must be second order. To determine them explicitly in the neighborhood of P_0, we rewrite Eq. (3.6) as

$$\frac{\partial \mathcal{P}^x_a}{\partial c^{b\mathcal{B}v}} = -f^{abc}c^{c\mathcal{B}u}\eta^x_{uv} + \mathcal{O}((q-q_0)^2), \qquad \frac{\partial \mathcal{P}^x_a}{\partial q^\alpha} = \mathcal{O}((q-q_0)^2), \tag{3.24}$$

whose solution is

$$\mathcal{P}^x_a = \frac{1}{2}\,f^{abc}\,c^{b\mathcal{A}u}c^{c\mathcal{A}v}\eta^x_{uv} + \mathcal{O}((q-q_0)^3). \tag{3.25}$$

Effective action: We are now equipped to determine the potential $\mathcal{V}$ of Eq. (3.4) in the vicinity of P_0. As follows from Eqs (3.10), (3.8), (3.23), the first three terms in $\mathcal{V}$, which are positive, are of order four in $(X - X_0)$ or $(q - q_0)$, while the last one, which is negative, is of order six and can be ignored. Introducing the $SU(2)_R$-doublets and notations

$$\mathscr{C}^{a\mathcal{A}} = \begin{pmatrix} i(c^{a\mathcal{A}1} + ic^{a\mathcal{A}2}) \\ (c^{a\mathcal{A}3} + ic^{a\mathcal{A}4})^* \end{pmatrix}, \qquad X \equiv X^a\,T^a, \quad \bar{X} \equiv X^a\,T^a, \quad c^{\mathcal{A}u} \equiv c^{a\mathcal{A}u}\,T^a, \tag{3.26}$$

the tree level potential can be written as

$$\begin{aligned}
&\mathcal{V} = e^{\mathcal{K}^{(0)}}\Big(l^2[X,\bar{X}]^a[X,\bar{X}]^a + 2[X,c^{\mathcal{A}u}]^a[c^{\mathcal{A}u},\bar{X}]^a + \frac{1}{4l^2}D^{ax}D^{ax}\Big) + \cdots\\
&\text{where} \qquad D^{ax} \equiv -if^{abc}\,\mathscr{C}^{b\mathcal{A}\dagger}\sigma^x\mathscr{C}^{c\mathcal{A}}, \qquad [\mathcal{X},\mathcal{Y}]^a \equiv if^{abc}\,\mathcal{X}^b\mathcal{Y}^c,
\end{aligned} \tag{3.27}$$

[13] $\mathcal{O}(q - q_0)$ denotes terms of order $c^{a\mathcal{A}u}$ or $(q^\alpha - q_0^\alpha)$.

with the dots standing for order five terms.

Collecting the leading contributions to the kinetic terms, the low energy effective action (3.2) of the type IIA theory at finite temperature and compactified on M or M'', close to their extremal transition, is at tree level and lowest order in scalar fields,

$$S_{\text{tree}} = \int d^4x\,\sqrt{-g}\,\Big\{\frac{\mathcal{R}}{2} - l^2\,\nabla_\mu X^a\nabla^\mu \bar{X}^a - g^{(0)}_{p\bar{q}}\,\partial_\mu X^p\partial^\mu \bar{X}^q - \frac{1}{2}\,\nabla_\mu c^{a\mathcal{A}u}\nabla^\mu c^{a\mathcal{A}u} - h^{(0)}_{\alpha\beta}\,\partial_\mu q^\alpha\partial^\mu q^\beta$$
$$- e^{\mathcal{K}^{(0)}}\Big(l^2[X,\bar{X}]^a[X,\bar{X}]^a + 2[X,c^{\mathcal{A}u}]^a[c^{\mathcal{A}u},\bar{X}]^a + \frac{1}{4l^2}D^{ax}D^{ax}\Big) + \cdots\Big\}. \tag{3.28}$$

Therefore, up to the "spectator multiplets", the Lagrangian density has the form of a minimally coupled rigid $\mathcal{N}=2$ supersymmetric $SU(N)$ gauge theory coupled to g hypermultiplets in the adjoint representation, formally coupled to gravity.

3.3 Tree level masses

The tree level potential (3.27) admits flat directions, along which masses for the degrees of freedom involved in the $SU(N)$ gauge theory are generated. Since they depend on the moduli, the one-loop free energy we will take into account in the following Sections will behave as an effective potential, able to lift the flat directions of the vector and hypermultiplet scalars charged under $SU(N)$. Actually, rather than the masses themselves, this goal requires the sum of the tree level squared masses of the bosonic degrees of freedom we now determine.

Let us start by characterizing the flat directions of $\mathcal{V}$. In the vicinity of P_0, whose coordinates in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$ are $(X^a_0 = 0, X^p_0; c^{a\mathcal{A}u}_0 = 0, q^\alpha_0)$, the set of vacua splits into a Cartesian product between:

- The space parameterized by the "spectator moduli" X^p and q^α, which are coordinates along the non-Abelian locus.
- The space of configurations of the X^a's and $c^{a\mathcal{A}u}$'s, which are canceling the semi-definite positive potential $\mathcal{V}$. It is defined by the conditions

$$[X,\bar{X}]=0, \qquad \forall \mathcal{A},u:\ \ [X,c^{\mathcal{A}u}]=0 \qquad \text{and} \qquad \forall a,x:\ \ D^{ax}=0, \tag{3.29}$$

  which admits Coulomb and Higgs branches to be specified later.

To determine the mass terms, we substitute in the action (3.28)

$$X^a \to X^a + \delta X^a, \qquad c^{a\mathcal{A}u} \to c^{a\mathcal{A}u} + \delta c^{a\mathcal{A}u}, \tag{3.30}$$

where $(X^a, c^{a\mathcal{A}u})$ satisfies the constraints (3.29), and focus on the quadratic terms of the potential. For $\delta X^b \delta \bar{X}^c$, we obtain from the first two terms of $\mathcal{V}$ in Eq. (3.27)

$$e^{\mathcal{K}^{(0)}}\Big(2l^2[\delta X, \delta\bar{X}]^a[X, \delta\bar{X}]^a + 2[\delta X, c^{\mathcal{A}u}]^a[c^{\mathcal{A}u}, \delta\bar{X}]^a\Big) := E_{bc}\, \delta X^b \delta \bar{X}^c, \tag{3.31}$$

while the contributions in $\delta X^b \delta X^c$ and their complex conjugate are irrelevant to compute the trace of the squared mass operator. Taking into account the normalization of the kinetic terms of the δX^a's in the action (3.28), we find a contribution to the trace

$$\operatorname{tr} M^2\big|_{\delta X} = \frac{2}{l^2}\, E_{aa} + \cdots = 4Ne^{\mathcal{K}^{(0)}}\Big(X^a\bar{X}^a + \frac{1}{l^2}\, c^{a\mathcal{A}u}c^{a\mathcal{A}u}\Big) + \cdots . \tag{3.32}$$

The factor 2 counts the real and imaginary parts of δX^a, the factor N arises from the relation $\operatorname{tr}(T^aT^b) = N\delta^{ab}$ and the ellipsis stand for higher order terms in the scalar fields.

Terms in $\delta c^{a\mathcal{A}u}\delta c^{b\mathcal{B}v}$ arise from the second term in (3.27),

$$e^{\mathcal{K}^{(0)}} 2[X, \delta c^{\mathcal{A}u}]^a[\delta c^{\mathcal{A}u}, \bar{X}]^a := \frac{1}{2}\, L_{a\mathcal{A}u, b\mathcal{B}v}\, \delta c^{a\mathcal{A}u}\delta c^{b\mathcal{B}v}, \tag{3.33}$$

and the D-terms. To evaluate the latter, it is convenient to rewrite the third term of Eq. (3.4) around P_0 in an alternative form, using Eq. (A.5),

$$e^{\mathcal{K}^{(0)}}\frac{1}{4l^2}\, D^{ax}D^{ax} = e^{\mathcal{K}^{(0)}}\frac{1}{4l^2}\Big(2[c^{\mathcal{A}v}, c^{\mathcal{A}u}]^a[c^{\mathcal{B}u}, c^{\mathcal{B}v}]^a - \epsilon_{uvu'v'}[c^{\mathcal{A}u}, c^{\mathcal{A}v}]^a[c^{\mathcal{B}u'}, c^{\mathcal{B}v'}]^a\Big). \tag{3.34}$$

Substituting (3.30), the only terms that may contribute to the trace of squared masses are

$$e^{\mathcal{K}^{(0)}}\frac{1}{l^2}\Big([\delta c^{\mathcal{A}v}, c^{\mathcal{A}u}]^a[\delta c^{\mathcal{B}u}, c^{\mathcal{B}v}]^a + [\delta c^{\mathcal{A}v}, c^{\mathcal{A}u}]^a[c^{\mathcal{B}u}, \delta c^{\mathcal{B}v}]^a\Big) := \frac{1}{2}\, P_{a\mathcal{A}u, b\mathcal{B}v}\, \delta c^{a\mathcal{A}u}\delta c^{b\mathcal{B}v}. \tag{3.35}$$

The kinetic terms of the $\delta c^{a\mathcal{A}u}$'s being canonical, we obtain

$$\operatorname{tr} M^2\big|_{\delta c} = (L+P)_{a\mathcal{A}u, a\mathcal{A}u} + \cdots = 4Ne^{\mathcal{K}^{(0)}}\Big(4gX^a\bar{X}^a + \frac{3}{2l^2}\, c^{a\mathcal{A}u}c^{a\mathcal{A}u}\Big) + \cdots . \tag{3.36}$$

All terms in $\delta X \delta c^{a\mathcal{A}u}$ or their complex conjugate do not contribute the trace. This completes the analysis arising from the scalar fields.

To proceed, we need the contribution associated to the vector bosons. The Yang-Mills Lagrangian density implicit in Eq. (3.28) can be written as

$$-\frac{1}{4}\, \tau^{(0)}_{AB}\, F^A_{\mu\nu}F^{B\mu\nu}, \tag{3.37}$$

where the index A takes values 0 or I to account for the graviphoton and $\tau^{(0)}_{AB}$ are the gauge kinetic functions evaluated at P_0 and $F^A = dA^A$. When (3.30) is applied and we keep the lowest order contribution in scalar fields, a mass matrix Q_{ab} is generated by the covariant derivatives and we obtain in flat Minkowski space,

$$\frac{1}{2}\,\tau^{(0)}_{AB}\,A^{A\mu}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^{B\nu} - \frac{1}{2}\,A^{a\mu}\,\eta_{\mu\nu}Q_{ab}\,A^{b\nu}. \tag{3.38}$$

Therefore, the contribution of the vector bosons to the trace of squared masses is

$$\operatorname{tr} M^2\big|_{A^\mu} = 3\,\tau^{(0)ba}Q_{ab} + \cdots = 6\,\tau^{(0)ba}f^{cda}f^{ceb}\Big(l^2X^d\bar{X}^e + \frac{1}{2}\,c^{d\mathcal{A}u}c^{e\mathcal{A}u}\Big) + \cdots\,, \tag{3.39}$$

where the factor 3 counts the number of degrees of freedom in massive spin-one fields. It is certainly possible to evaluate the inverse metric $\tau^{(0)ab}$ from its form dictated by $\mathcal{N}=2$ supergravity [23]. However, this can be avoided by noticing that $\operatorname{tr} M^2|_{A^\mu}$ is known when $g=1$. In this conformal case, there is no Higgs branch and, in the Coulomb phase, all massive degrees of freedom sit in long vector multiplets (see Table 2). Each of them contains 3 degrees of freedom associated to the vector bosons and 5 from the scalars, which are all degenerate. Therefore, we have

$$\operatorname{tr} M^2\big|^{g=1}_{A^\mu} = \frac{3}{5}\Big(\operatorname{tr} M^2\big|^{g=1}_{\delta X} + \operatorname{tr} M^2\big|^{g=1}_{\delta c}\Big) + \cdots = 12Ne^{\mathcal{K}^{(0)}}\Big(X^a\bar{X}^a + \frac{1}{2l^2}\,c^{a1u}c^{a1u}\Big) + \cdots\,. \tag{3.40}$$

Comparing with Eq. (3.39), we find $\tau^{(0)ba}f^{cda}f^{ceb}l^2 \equiv 2Ne^{\mathcal{K}^{(0)}}\delta^{de}$ and therefore for arbitrary $g \geq 1$,

$$\operatorname{tr} M^2\big|_{A^\mu} = 12Ne^{\mathcal{K}^{(0)}}\Big(X^a\bar{X}^a + \frac{1}{2l^2}\,c^{a\mathcal{A}u}c^{a\mathcal{A}u}\Big) + \cdots\,. \tag{3.41}$$

Collecting the contributions of the scalar masses Eqs (3.32), (3.36) and spin-one fields Eq. (3.41), we arrive at the final result for the trace of the squared mass operator in the bosonic sector of the $SU(N)$ gauge theory coupled to g hypermultiplets in the adjoint representation, valid in the vicinity of P_0,

$$\operatorname{tr} M^2\big|_{\text{gauge}} = 16Ne^{\mathcal{K}^{(0)}}\Big((g+1)X^a\bar{X}^a + \frac{1}{l^2}\,c^{a\mathcal{A}u}c^{a\mathcal{A}u}\Big) + \cdots\,. \tag{3.42}$$

We recall that the above expression applies to any scalar configuration satisfying (3.29), in which case all tadpoles are vanishing.

3.4 Lifting the Coulomb branch at one-loop

The space of classical flat directions around P_0 splits into several branches. The Coulomb phase corresponds to scalar VEV's such that the matrices X^aT^a and $c^{a\mathcal{A}u}T^a$ sit in the Cartan sub-algebra. In this case, the two first conditions in Eq. (3.29) are satisfied, while the D-term condition follows from Eq. (3.34). Denoting as T^i ($i = 1, \dots, N-1$) the Cartan generators of $SU(N)$ and $T^{\hat a}$ ($\hat a = N, \dots, N^2-N$) the remaining ones, we have

$$\text{Coulomb branch: } \Big\{\big(X^i \text{ arbitrary}, X^{\hat a} = 0, c^{i\mathcal{A}u} \text{ arbitrary}, c^{\hat a\mathcal{A}u} = 0\big)\Big\} \times \Big\{\big(X^p, q^\alpha\big) \text{ arbitrary}\Big\}, \tag{3.43}$$

which corresponds to a compactification on the CY space M. As noticed in Footnote 12, M is a smooth manifold, except when some $X^i = X^j$ and $c^{i\mathcal{A}u} = c^{j\mathcal{A}u}$ for $i \neq j$, so that $SU(N)$ is broken to a non-Abelian subgroup of rank $N-1$.

In the coulomb phase, when the X^i's are generic but $c^{i\mathcal{A}u} = 0$ (for all $\mathcal{A}$ and u), no mass mixing terms of the form $\delta X^a \delta c^{b\mathcal{B}v}$ or $\delta \bar X^a \delta c^{b\mathcal{B}v}$ are generated by the shift (3.30) in $\mathcal{V}$. Therefore, the N^2-N gauge bosons that are acquiring masses eat half of the degrees of freedom of the $\delta X^{\hat a}$'s and $\delta \bar X^{\hat a}$'s, and combine into N^2-N short massive vector multiplets. Thus, the ratio $\operatorname{tr} M^2|_{A^\mu} / \operatorname{tr} M^2|_{\delta X}$ must equal $3/(2-1) = 3$, which is satisfied by Eqs (3.39) and (3.32). The complete superfield content in this case is reported in Table 2.

Similarly, when $X^i = 0$ (for all i) but the $c^{i\mathcal{A}u}$'s are generic, there are still no mixing terms between the vector and matter scalars. However, the N^2-N vector boson eat as many would-be-Goldstone bosons now among the $\delta c^{\hat a\mathcal{A}u}$, and the massive spectrum contains N^2-N long vector multiplets. Thus, the ratio $\operatorname{tr} M^2|_{A^\mu} / \operatorname{tr} M^2|_{\delta X}$ must equal $3/2$, again satisfied by Eqs (3.39) and (3.32).

On the contrary, at a generic point in the branch (3.43), the mass mixing terms imply the mass eigenstates are combinations of vector and hypermultiplet scalars. Therefore, $\operatorname{tr} M^2|_{\delta X}$ and $\operatorname{tr} M^2|_{\delta c}$ cannot be interpreted as sums of squared masses in separated vector and hypermultiplet scalar sectors. Only the sum $\operatorname{tr} M^2|_{\delta X} + \operatorname{tr} M^2|_{\delta c}$ makes sense, as the contribution of the full spin-zero sector. Moreover, the would-be-Goldstone bosons are combinations of the non-Cartan vector and hypermultiplet scalars.

Restricted to the weak string coupling regime, we are now ready to write the one-loop thermal effective action. In the Coulomb branch, it amounts to adding the tree level action

	Scalars acquiring VEV's		Superfields				
	In vector multiplets	In hyper-multiplets	Vector multiplets			Hypermultiplets	
			Massless (moduli)	Massive short	Massive long	Massless (moduli)	Massive
Coulomb phase	X^i	none	$N-1$	N^2-N	0	$g(N-1)$	$g(N^2-N)$
	X^i or none	$c^{i\mathcal{A}u}$	$N-1$	0	N^2-N	$g(N-1)$	$(g-1)(N^2-N)$
Higgs phase	none	$\mathscr{C}^{a\mathcal{A}}$ mod. gauge orbits such that $D^{ax}=0$	0	0	N^2-1	$(g-1)(N^2-1)$	0

Table 2: Superfield contents in the Coulomb and Higgs branches (when $g \geq 2$) associated to the $\mathcal{N}=2$ $SU(N)$ gauge theory coupled to g hypermultiplets in the adjoint representation, which is encountered in the neighborhood of a non-Abelian locus in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$. The scalars X^p and q^α of the massless spectator vector multiplets and hypermultiplets are not represented. At special loci in the Coulomb branch, where some $X^i = X^j$ and $c^{i\mathcal{A}u} = c^{j\mathcal{A}u}$ for $i \neq j$, some generically massive multiplets are actually massless, and the $SU(N)$ gauge symmetry is broken to a non-Abelian subgroup of rank $N-1$, rather than $U(1)^{N-1}$.

(3.28) in a vacuum (3.43) to the one-loop Coleman-Weinberg effective potential $\mathcal{F}$,

$$S_{\text{1-loop}} = \int d^4x\sqrt{-g}\left\{\frac{\mathcal{R}}{2} - l^2\partial_\mu X^i\partial^\mu\bar{X}^j - g^{(0)}_{p\bar{q}}\,\partial_\mu X^p\partial^\mu\bar{X}^q - \frac{1}{2}\partial_\mu c^{a\mathcal{A}u}\partial^\mu c^{a\mathcal{A}u} - h^{(0)}_{\alpha\beta}\partial_\mu q^\alpha\partial^\mu q^\beta - \mathcal{F}\right\}. \tag{3.44}$$

As seen in Eq. (2.21), $\mathcal{F}$ is actually the free energy density, which in the present case is

$$\mathcal{F} = -T^4\left\{\Big(4+4h_{11}+4(h_{12}+1)\Big)G(0) + \sum_{\hat{s}} G\Big(\frac{M_{\hat{s}}}{T}\Big) + \mathcal{O}\big(e^{-\frac{M_{\rm min}}{T}}\big)\right\}, \tag{3.45}$$

where the index $\hat{s}$ labels all pairs of degenerate boson/fermion states in the massive vector multiplets and hypermultiplets involved in the $SU(N)$ gauge theory and collected in Table 2. In Eq. (3.45), we take the temperature to be below the lower bound $M_{\rm min} > 0$ at P_0 of the remaining masses of the full string spectrum. $\mathcal{F}$ is minimal when and only when all classical masses in the $SU(N)$ gauge sector vanish, $\forall \hat{s}:\ M_{\hat{s}} = 0$. Using the general formula (3.42) in the Coulomb branch,

$$\sum_{\hat{s}} M^2_{\hat{s}} \equiv \operatorname{tr} M^2\big|_{\rm gauge} = 16Ne^{\mathcal{K}^{(0)}}\Big((g+1)X^i\bar{X}^i + \frac{1}{l^2}\,c^{i\mathcal{A}u}c^{i\mathcal{A}u}\Big) + \cdots, \tag{3.46}$$

this implies $X^i = 0$, $c^{i\mathcal{A}u} = 0$. Therefore, all moduli involved in the Coulomb phase of the $SU(N)$ gauge theory are lifted. Their kinetic terms being diagonal, the masses they acquire at one-loop are

$$M^2_{i\text{1-loop}} = \frac{1}{l^2}\,\frac{\partial^2\mathcal{F}}{\partial X^i\partial X^i}\bigg|_{X^j=c^{j\mathcal{A}u}=0} = \frac{T^2}{16}\,\frac{1}{l^2}\sum_s \frac{\partial^2 M_s^2}{\partial X^i\partial X^i}\bigg|_{X^j=c^{j\mathcal{A}u}=0} = T^2\,(g+1)\,\frac{N}{l^2}e^{\mathcal{K}^{(0)}},$$

$$M^2_{i\mathcal{A}u\text{1-loop}} = \frac{\partial^2\mathcal{F}}{\partial c^{i\mathcal{A}u}\partial c^{i\mathcal{A}u}}\bigg|_{X^j=c^{j\mathcal{A}u}=0} = \frac{T^2}{16}\sum_s \frac{\partial^2 M_s^2}{\partial c^{i\mathcal{A}u}\partial c^{i\mathcal{A}u}}\bigg|_{X^j=c^{j\mathcal{A}u}=0} = T^2\,2\,\frac{N}{l^2}e^{\mathcal{K}^{(0)}}, \quad (3.47)$$

while the classically massless $U(1)^{N-1}$ spin-1 fields do not acquire masses. Moreover, due to the arbitrariness in the choice of Cartan subalgebra at the origin of the Coulomb branch, we conclude that all vector multiplet and hypermultiplet scalars X^a and $c^{a\mathcal{A}u}$, even though classically massless, have one-loop masses at their point of stabilization,

$$M^2_{X\text{1-loop}} = T^2\,(g+1)\,\frac{N}{l^2}e^{\mathcal{K}^{(0)}}, \qquad M^2_{c\text{1-loop}} = T^2\,2\,\frac{N}{l^2}e^{\mathcal{K}^{(0)}}, \quad (3.48)$$

where the full $SU(N)\times U(1)^{h_{11}-(N-1)+1}$ gauge symmetry is restored.

In a homogeneous and isotropic universe, the free energy density (3.45) leads along the $SU(N)$ non-Abelian locus to the state equation for radiation, $\rho = 3P$, when T is low enough. Consequences of this fact are similar to those encountered in the case of the conifold locus, below Eq. (2.30). A cosmological evolution for the scale factor and temperature exists, with static scalars fields,

$$a(t)\propto \frac{1}{T(t)}\propto \sqrt{t}\ , \quad X^a\equiv c^{a\mathcal{A}u}\equiv 0\ , \quad X^p, q^\alpha\ \text{constant}, \quad (3.49)$$

where t is the cosmological time. Since T decreases, the consistency of the approximation $T \ll M_{\rm min}$ used to neglect exponentially suppressed contributions in the free energy (3.45) is guaranteed. As in the $\mathcal{N}=4$ case ($g=1$ here) analyzed in Ref. [11], other time-evolutions compatible with homogeneity and isotropy exist, where the moduli X^i and $c^{i\mathcal{A}u}$ oscillate with damping in the Coulomb branch, thus converging to their minimum. The fact that their masses are of order the temperature scale and decreases as the universe expand implies the cosmological moduli problem is avoided.

3.5 Lifting the Higgs branch at one-loop

At the origin of the Coulomb branch, the conditions $D^{ax}=0$ become non-trivial constraints on the hypermultiplet scalars, which define the Higgs phase,

$$\text{Higgs branch}: \quad \Big\{\big(X^a=0, \mathscr{C}^{a\mathcal{A}} \text{ such that } D^{bx}=0\big)\Big\}\times\Big\{\big(X^p,q^\alpha\big)\text{ arbitrary}\Big\}. \quad (3.50)$$

The above conditions fix $3(N^2-1)$ components among the $4g(N^2-1)$ scalars $c^{a\mathcal{A}u}$. Along the flat directions, the $SU(N)$ local symmetry is completely Higgsed spontaneously. The remaining global $SU(N)$ orbits can be used to gauge away N^2-1 would-be-Goldstone bosons, so that $4(g-1)(N^2-1)$ flat directions of inequivalent vacua remain. By supersymmetry, the latter can be parameterized by the scalars of $(g-1)(N^2-1)$ neutral hypermultiplets. Thus, the above Higgs branch exists only for $g \geq 2$, in which case it is realized geometrically by compactifying on the CY M'' with Hodge numbers given in Eq. (3.1). Actually, N^2-1 of the initial $g(N^2-1)$ hypermultiplets combine with the Higgsed vector multiplets into N^2-1 massive long vector multiplets, as summarized in Table 2. Therefore $\operatorname{tr} M^2|_{A^\mu} \,/\, \operatorname{tr} M^2|_{\delta X} = \operatorname{tr} M^2|_{\delta c} \,/\, \operatorname{tr} M^2|_{\delta X} = 3/2$ must be satisfied, which is consistent with Eqs (3.32), (3.36) and (3.39) when $X^a = 0$.

In the case of the Coulomb phase, we introduced an arbitrary choice of Cartan generators T^i among the T^a's. In a similar way, we now define an arbitrary set of coordinates ξ^m $\big(m = 1, \dots, 4(g-1)(N^2-1)\big)$ along the Higgs phase flat directions. They satisfy $f^{abc}\,\mathscr{C}^{b\mathcal{A}\dagger}(\xi)\sigma^x\mathscr{C}^{c\mathcal{A}}(\xi) = 0$ and the Jacobian matrix $\left(\frac{\partial c^{a\mathcal{A}u}}{\partial \xi^m}\right)$ is of rank $4(g-1)(N^2-1)$. The origin of the Higgs branch is denoted ξ_0^m. In these notations, the one-loop effective action of the type IIA string theory compactified on M'' at finite temperature is, in the neighborhood of P_0,

$$S_{\text{1-loop}} = \int d^4x\,\sqrt{-g}\left\{\frac{\mathcal{R}}{2} - g^{(0)}_{p\bar q}\,\partial X^p\partial \bar X^q - h^{(0)}_{mn}\,\partial\xi^m\partial\xi^n - h^{(0)}_{\alpha\beta}\,\partial q^\alpha\partial q^\beta - \mathcal{F}\right\}, \tag{3.51}$$

where the induced metric of the ξ^m's is

$$h^{(0)}_{mn} = \frac{1}{2}\,\frac{\partial c^{a\mathcal{A}u}}{\partial \xi^m}\bigg|_{\xi_0}\,\frac{\partial c^{a\mathcal{A}u}}{\partial \xi^n}\bigg|_{\xi_0}. \tag{3.52}$$

In the present case, the free energy density $\mathcal{F}$ takes the form

$$\mathcal{F} = -T^4\left\{\Big(4 + 4h''_{11} + 4(h''_{12}+1)\Big)G(0) + 8\sum_a G\Big(\frac{M_a}{T}\Big) + \mathcal{O}\big(e^{-\frac{M_{\text{min}}}{T}}\big)\right\}, \tag{3.53}$$

where the factor 8 counts the number of boson/fermion pairs of states in the long vector multiplets of masses M_a ($a = 1, \dots, N^2-1$). The contributions of all the other massive modes of the spectrum are exponentially suppressed when $T < M_{\text{min}}$.

The minimum of $\mathcal{F}$ is reached when the classical masses M_a vanish. Applying Eq. (3.42) in the Higgs branch,

$$8\sum_a M_a^2 \equiv \mathrm{tr}\, M^2\big|_{\rm gauge} = 16\frac{N}{l^2}e^{\mathcal{K}^{(0)}}\, c^{a\mathcal{A}u}c^{a\mathcal{A}u} + \cdots\,, \tag{3.54}$$

we see that this selects the origin of the Higgs branch, $c^{a\mathcal{A}u} = 0$. Therefore, all classical flat directions ξ^m are lifted and admit a unique minimum at ξ_0^m at the one-loop level. The squared mass matrix of the ξ^m's is

$$\Lambda''^m{}_n = \frac{1}{2}h^{(0)ml}\,\frac{\partial^2\mathcal{F}}{\partial\xi^l\partial\xi^n}\bigg|_{\xi_0} = \frac{T^2}{16}\,\frac{1}{2}h^{(0)ml}\,8\sum_a\frac{\partial^2 M_a^2}{\partial\xi^l\partial\xi^n}\bigg|_{\xi_0} = T^2\,2\,\frac{N}{l^2}e^{\mathcal{K}^{(0)}}\delta^m_n, \tag{3.55}$$

where we have used the fact that $c^{a\mathcal{A}u}|_{\xi_0} = 0$ to reach the last equality. Thus, the ξ^m's are mass eigenstates and degenerate. Since the parametrization of the Higgs branch was chosen arbitrarily, we obtain that all scalars $c^{a\mathcal{A}u}$ acquire a common mass given by Eq. (3.55). Consistently, this is the result we already found by approaching the $SU(N)$ non-Abelian locus from the Coulomb branch, Eq. (3.48).

From a cosmological point of view, the expanding universe with static moduli and filled with radiation of Eq. (3.49) appears as a particular case of a second class of homogeneous and isotropic time-evolutions. In this class, even if the ξ^m's oscillate with damping, the cosmological moduli problem is avoided [11].

4 Stabilization at intersections of extremal transition loci

In the previous Sections, we have shown that the thermal effective potential admits local minima along submanifolds in moduli space, where the internal CY develops singularities. Therefore, the intersection points of various such loci are expected to define dynamically preferred configurations of the internal space. In the following, we illustrate this fact on an example in type IIA (IIB), where the Kähler (complex structure) moduli space is completely lifted, together with some of the complex structure (Kähler) moduli. This implies in particular that the axio-dilaton field of the heterotic dual description is stabilized. We then discuss how this phenomenon is expected to apply in generic CY compactifications to all vector multiplet and most of the hypermultiplet scalars.

4.1 Example

Let us consider an example, with small Hodge number h_{11} [28, 29]. The type IIA model we analyze at finite temperature is compactified on a CY manifold M obtained by resolving the singularities of a degree 12 hypersurface in $\mathbb{P}^4_{(1,1,2,2,6)}$. Denoting the projective coordinates as $x_1, \dots, x_5$, the ambient space presents initially a singularity of type A_1 at $x_1 = x_2 = 0$. Along this locus, the polynomial of M defines a curve $\mathcal{C}$ of genus 2. Blowing up the ambient space singularity, M becomes a smooth CY manifold, whose Kähler moduli space $\mathcal{M}_V$ admits a non-Abelian locus with $N = 2$ and $g = 2$. Counting the allowed monomials of the defining polynomial of M, one finds there are $h_{12} = 128$ complex structure moduli.

Equivalently, the model can be analyzed in type IIB compactified on the mirror CY three-fold W. The latter is defined by the vanishing locus of degree 12 polynomials in $\mathbb{P}^4_{(1,1,2,2,6)}$, modded out by some particular $\mathbb{Z}_6^2 \times \mathbb{Z}_2$ group. The most general hypersurface consistent with this action is [13, 14, 28]

$$\mathscr{P} = x_1^{12} + x_2^{12} + x_3^6 + x_4^6 + x_5^2 - 12\psi\, x_1x_2x_3x_4x_5 - 2\phi\, x_1^6 x_2^6 \,, \tag{4.1}$$

which admits 2 complex structure deformations, ψ and ϕ. Therefore, the manifold M admits $h_{11} = 2$ Kähler moduli. Defining

$$z_1 = -\frac{1}{864}\frac{\phi}{\psi^6}\,, \qquad z_2 = \frac{1}{\phi^2}\,, \tag{4.2}$$

the locus $\mathscr{P} = 0$ develops singularities when $\Delta_c\Delta_{nA}$ vanishes, where

$$\Delta_c \equiv (1 - z_1)^2 - z_1^2 z_2\,, \qquad \Delta_{nA} \equiv 1 - z_2\,. \tag{4.3}$$

When $\Delta_c = 0$, nodes are occurring, which are identified under $\mathbb{Z}_6^2 \times \mathbb{Z}_2$. Therefore, $\mathcal{M}_V$ admits a conifold locus characterized by $R = S = 1$.[14] When $\Delta_{nA} = 0$, other isolated singularities occur, yielding again a single singular point on the quotient. Since we know $\mathcal{M}_V$ admits a non-Abelian locus, this second point-like singularity must be associated to the type IIB realization of the $SU(2)$ gauge theory. This is confirmed once the orbifold singularities implied by the discrete modding are blown up to obtain a smooth manifold W [28, 29].

[14] We use the fact that the mirror of the conifold W is a conifold M.

Since $R = S$, there is no extremal transition associated to the conifold locus. On the contrary, when M develops the genus-g curve of A_1 singularities, the fact that $g > 1$ implies that M can be deformed into a distinct smooth CY M''. The ambient spaces, degrees of polynomials and Hodge numbers of the families of CY manifolds on either side of the associated non-Abelian extremal transition are [13, 14]

$$\mathbb{P}^4_{(1,1,2,2,6)}[12](2,128) \longleftrightarrow \mathbb{P}^5_{(1,1,1,1,1,3)}[2,6](1,129). \tag{4.4}$$

Beside the $U(1)_{\rm grav}$ gauge factor associated to the graviphoton, the type IIA compactifications on M and M'' realize geometrically the phases of the $U(1)_{\rm con}$ Abelian theory coupled to a charged hypermultiplet and $SU(2)$ super-Yang-Mills theory coupled to two hypermultiplets in the adjoint representation,

$$\begin{aligned} &\text{type IIA on } M \ : && \Big\{\text{Coulomb phase of } U(1)_{\rm con}\Big\} \times \Big\{\text{Coulomb phase of } SU(2)\Big\} \\ &\text{type IIA on } M'' : && \Big\{\text{Coulomb phase of } U(1)_{\rm con}\Big\} \times \Big\{\text{Higgs phase of } SU(2)\Big\}. \end{aligned} \tag{4.5}$$

The conifold and non-Abelian loci intersect at two points on the compactified moduli space $\mathcal{M}_V$ [28],

$$(z_1, z_2) = (1/2, 1) \quad \text{or} \quad (\infty, 1). \tag{4.6}$$

In either of these configurations, the node and isolated singularity on the hypersurface $\mathscr{P} = 0$ are separated from each other and generate independent massless states. From the type IIA point of view on the singular space M, the single massless black hole hypermultiplet has charges $Q_1 = 1$ with respect to $U(1)_{\rm con}$ and Q_2 with respect to the $U(1)$ Cartan subgroup of $SU(2)$. Similarly, the vector multiplet and $g = 2$ hypermultiplets in the adjoint representation of $SU(2)$ are neutral with respect to $U(1)_{\rm con}$. Therefore, we can combine the results of the previous Sections and consider the extended moduli space $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$, which takes into account the scalar fields of the light non-perturbative states arising in the vicinity of the points (4.6) in $\mathcal{M}_V$. Choosing a point P_0 located at the intersection of the conifold and non-Abelian loci in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$, the tree level low energy effective action of the type IIA string theory compactified on M or M'' takes near P_0 the form,

$$S_{\rm tree} = \int d^4x\,\sqrt{-g}\,\Big\{\frac{\mathcal{R}}{2} - l_1^2\,\partial_\mu X^1\partial^\mu \bar X^1 - l^2\,\nabla_\mu X^a\nabla^\mu \bar X^a$$
$$- \frac{1}{2}\,\nabla_\mu c^{1u}\nabla^\mu c^{1u} - \frac{1}{2}\,\nabla_\mu c^{a\mathcal{A}u}\nabla^\mu c^{a\mathcal{A}u} - h^{(0)}_{\alpha\beta}\,\partial_\mu q^\alpha\partial^\mu q^\beta$$
$$- e^{\mathcal{K}^{(0)}}\Big(2\,|X^1|^2\,c^{1u}c^{1u} + \frac{1}{4l_1^2}\,(c^{1u}c^{1u})^2\Big)$$
$$- e^{\mathcal{K}^{(0)}}\Big(l^2[X,\bar X]^a[X,\bar X]^a + 2[X,c^{\mathcal{A}u}]^a[c^{\mathcal{A}u},\bar X]^a + \frac{1}{4l^2}D^{ax}D^{ax}\Big) + \cdots\Big\}. \tag{4.7}$$

Our conventions are as follows: X^1 is the scalar partner of the $U(1)_{\rm con}$ gauge boson and X^a ($a=2,3,4$) is in the adjoint of $SU(2)$. Similarly, c^{1u} are the components of the black hole hypermultiplet, while $c^{a\mathcal{A}u}$ are those of the two hypermultiplets in the adjoint of $SU(2)$. The scalars of the 127 (see below for the counting) hypermultiplets that are neutral with respect to $U(1)_{\rm con}\times SU(2)$ are denoted q^α and the metric in their subspace in $\tilde{\mathcal{M}}_H$ at P_0 is $h^{(0)}_{\alpha\beta}$. Similarly, l_1^2, l^2 are the non-vanishing entries of the Kähler metric on $\tilde{\mathcal{M}}_V$ at P_0, whose coordinates are $(X^1_0=0, X^a_0=0; c^{1u}_0=0, c^{a\mathcal{A}u}_0=0, q^\alpha_0)$, so that the Kähler potential defined in Eq. (2.5) reduces to $\mathcal{K}^{(0)}=-\ln[i(F_0-\bar F_0)]$.

Taking into account one-loop corrections, the scalars X^1, X^a, c^{1u} and $c^{a\mathcal{A}u}$ are stabilized at zero and acquire masses of order the temperature scale, while the q^α's remain flat directions of the thermal effective potential. Moreover, the full $U(1)_{\rm grav}\times U(1)_{\rm con}\times SU(2)$ gauge theory is restored. From a geometrical point of view, starting from a type IIA compactification on M, the quantum/thermal effects on the perturbative moduli imply:

- The $h_{11}=2$ Kähler moduli are stabilized in one of the two minima given in Eq. (4.6).

- The scalars of $g(N-1)=2$ hypermultiplets are stabilized at the origin of the Coulomb branch of $SU(2)$ in $\mathcal{M}_H$.

- The scalars of the $h_{12}+1-g(N-1)=127$ left-over hypermultiplets remain flat directions in $\mathcal{M}_H$.

Similarly, starting from a type IIA compactification on M'', the thermal free energy density implies:

- The $h''_{11}=1$ complexified Kähler modulus parameterizing $\mathcal{M}''_V$ is stabilized.

- The scalars of $(g-1)(N^2-1)=3$ hypermultiplets are stabilized at the origin of the

Higgs branch of $SU(2)$ in $\mathcal{M}''_H$.

• The scalars of the $h''_{12}+1-(g-1)(N^2-1)=127$ left-over hypermultiplets remain flat directions in $\mathcal{M}''_H$.

The heterotic dual: At zero temperature, the type IIA string model compactified on M admits a heterotic dual description [20]. This follows from the fact that the CY manifolds in the family $\mathbb{P}^4_{(1,1,2,2,6)}[12]$ are $K3$-fibrations [19]. The heterotic model is compactified on $K3\times T^2$, where the 2-torus moduli T_h and U_h are identified, $T_h\equiv U_h$ (their difference being projected out), and the full non-Abelian gauge group is Higgsed. Consistently, the massless spectrum contains 2 vector multiplets associated to the T_h and S_h moduli, where S_h is the heterotic axio-dilaton, together with 129 neutral hypermultiplets.

T_h and S_h are special coordinates that can be identified with those obtained by inverting the mirror map: $z_1(t_1,t_2)$, $z_2(t_1,t_2)$. To render the identification precise [20], one observes that in the large complex structure limit of W, $t_2\to+\infty$, one finds

$$z_1=\frac{1728}{j(t_1)}+\cdots\,,\qquad z_2=e^{-t_2}+\cdots\,,\tag{4.8}$$

where j is the $SL(2,\mathbb{Z})$-invariant modular form. Therefore, $z_2\to 0$ and the two roots of the discriminant locus Δ_c in Eq. (4.3) merge into $z_1=1$. These facts match exactly the behavior of the perturbative heterotic model, under the identification

$$T_h\equiv t_1,\qquad S_h\equiv t_2.\tag{4.9}$$

Actually, the latter develops an $SU(2)$ enhanced gauge symmetry[15], when $T_h=i$ modulo the classical T-duality group $SL(2,\mathbb{Z})$, in perfect agreement with Eq. (4.8) for $z_1=1$. Moreover, when t_2 is finite, the conifold locus splits into two branches, as predicted by the exact pure $SU(2)$ $\mathcal{N}=2$ super-Yang-Mills theory [30]. Being asymptotically free, the latter reduces in the IR to a $U(1)$ gauge theory coupled to a single (dyonic) hypermultiplet, realized as $U(1)_{\rm con}$ in the type II setup [21, 22].

In their exact versions, the type II and heterotic models are supposed to be equivalent. Therefore, switching on finite temperature on both theories must lead to a new dual pair of non-supersymmetric models. This expectation is confirmed by the fact that at the levels of the worldsheet conformal field theories, finite temperature is introduced by implementing

[15] To not be confused with the $SU(2)$ gauge group occurring at the type II non-Abelian locus.

spontaneous breakings of supersymmetry "à la Scherk-Schwarz", along the Euclidean time circles. Using an adiabatic argument [31], under a free action, the two theories remain dual. Therefore, the stabilization of the complex structure moduli z_1, z_2 of W at one of the two points in Eq. (4.6) translates immediately into a stabilization of the torus modulus T_h and axio-dilaton S_h in the dual heterotic model at finite temperature. The latter are given by the inverse mirror map, $T_h(z_1, z_2)$, $S_h(z_1, z_2)$, where $z_1 = 1/2$ or ∞ and $z_2 = 1$. As seen in Eq. (4.8), the obtained value of S_h corresponds to a strong coupling regime of the heterotic theory.

Actually, the two local minima of (T_h, S_h) are uniquely determined, modulo the orbit of the exact heterotic duality group. Since the complex structure moduli space of W is exactly known at tree level in type IIB, the exact heterotic duality group is nothing but the monodromy group derived around the singular loci of the type IIB complex structure moduli space $\mathcal{M}_V$. As shown in Ref. [22], the latter contains the perturbative heterotic duality group (including the quantized axionic shift), the monodromies of the exact pure $SU(2)$ $\mathcal{N} = 2$ super-Yang-Mills theory, as well as a generator associated to the non-Abelian locus in $\mathcal{M}_V$ that exchanges roughly T_h with S_h [19, 22].

From the heterotic viewpoint, the origin of the $SU(2)$ gauge theory coupled to two adjoint hypermultiplets at $z_2 = 1$ is intrinsically non-perturbative and may be related to the existence of the NS5-brane. In fact, translated via S-duality into a type I picture [32], NS5-branes would be mapped into D5-branes that may play a role analogous to that of D1-branes already considered in Ref. [11]. There, D1-brane states winding internal 1-cycles were taken into account in the evaluation of the thermal free energy, whose effect was to stabilize internal moduli. Adding the contributions of D5-brane states winding internal 5-cycles in the evaluation of the free energy may lead to a stabilization of the type I dilaton. Alternatively, the contributions of the solitonic D1-brane states were shown to be equivalently described in terms of E1-instantons wrapping the Euclidean time circle $S^1(R_0)$ and internal 1-cycles. Thus, it would be interesting to see if E5-instantons wrapping $S^1(R_0)$ and internal 5-cycles would contribute in such a way to generate a potential for the type I dilaton.

Finally, note that the depth of the minimum of the free energy density depends only on the number of classically massless states at this point. Therefore, the two minima in Eq. (4.6) are degenerate. Moreover, both are at finite distance in the compactified moduli space

$\mathcal{M}_V$. Therefore, it may be interesting to find instantonic transitions between them, and analyze resulting physical consequences.

4.2 Discussion

The qualitative behavior and stabilization issues of the example of compactification we have analyzed are shared by numerous models based on other families of threefolds, with small Hodge numbers h_{11}. For instance, cases where $h_{11} = 2$ or 3, $N = 2$ or 3 and $g = 2, \ldots, 15$ can be found in Refs [13, 14].

In fact, in any type II compactification on a CY space, we expect the vector multiplet moduli space $\mathcal{M}_V$ to be completely lifted once finite temperature is switched on, the latter point being certainly relevant to describe the cosmological evolution of our universe. Geometrically, this means that all Kähler moduli in type IIA and all complex structure moduli in type IIB have masses of order the temperature scale. From the IIA point of view (and similarly in the IIB mirror picture), the mechanism is based on the fact that all homology classes of 2-cycles can vanish and that D2-branes wrapped on their representatives should always give rise to non-perturbative BPS states that are massless (at zero temperature), when the cycles collapse. In this work, this was analyzed in detail at conifold points, as well as at loci of $SU(N)$ enhanced gauge symmetries coupled to g hypermultiplets in the adjoint representations. It would be interesting to extend our approach to other points where 2-cycles are vanishing, by identifying the geometrically engineered gauge theories and the associated massless BPS states. For instance, one may analyze the case of non-Abelian gauge theories coupled to matter in the fundamental representations, which is considered in Ref. [15].

Flat directions in the classical hypermultiplet moduli space $\mathcal{M}_H$ are also lifted. This is the case for the directions that realize branches of the above mentioned gauge theories. In other words, say in type IIA, the 3-cycles that can be resolved into 2-cycles when they collapse are expected to be associated to quaternionic directions in $\mathcal{M}_H$ lifted by the thermal effective potential. For instance, these directions parameterize the Higgs branch arising at a conifold locus (when $R > S$), or the Coulomb or Higgs branches of the non-Abelian case we have analyzed. A question then arises: Can we stabilize this way all complex structure moduli in type IIA (Kähler moduli in type IIB)? To answer this question, let us consider a CY manifold M admitting 2-cycles that cannot be deformed into 3-cycles. Such a case was understood physically in our study or the conifold locus in type IIA, when $R = S$ and no Higgs branch exists. By mirror symmetry, there exist 3-cycles in the mirror CY W that cannot be resolved into 2-cycles. Utilizing W to compactify the type IIA string, the eventual (gauge) theory realized geometrically in the vicinity of the vanishing locus in $\mathcal{M}_H$ of these 3-cycles is not know to us. Therefore, we are not able to identify possible massless states occurring at these points, which would induce a local minimum of the free energy density and a stabilization of the associated complex structure moduli of W. Clearly, it would be very interesting to clarify this issue.

The above discussion of $\mathcal{M}_H$ concerns the h_{12} quaternionic directions associated to the complex structure of the internal space in type IIA. The remaining one, associated to the unique $(3,0)$-homology class, is parameterized by the scalars of the universal hypermultiplet, which contains the type II dilaton. Since the 3-cycles involved in the discussion of the tree level masses we considered can be resolved into 2-cycles, the universal hypermultiplet was always "spectator" and therefore unlifted by the thermal effective potential. This fact is actually consistent with our restriction to the case of a dilaton field sitting in a weak coupling regime, throughout the process of moduli stabilization. In fact, in string-frame, the one-loop correction to the vacuum energy is independent of the dilaton. In the Einstein frame, the vacuum energy acquires a dilaton dressing, which is however absorbed in the overall T^4 factor (see e.g. Eq. (2.23)), where the temperature measured in this frame is defined as

$$T = \frac{e^{\phi}}{2\pi R_0}. \tag{4.10}$$

Therefore, it is only by taking into account higher loop corrections and non-perturbative effects that the thermal effective potential would source the type II dilaton, though at strong coupling. It may then be possible to study this regime in the dual heterotic picture, where the hypermultiplet moduli space $\mathcal{M}_H$ is exactly know, given the fact that S_h sits in a vector multiplet. Working at heterotic weak coupling, the one-loop free energy evaluated on the heterotic side may stabilize the hypermultiplet containing the type II dilaton.

In this paper, the attraction to a point P_0 in $\tilde{\mathcal{M}}_V \times \tilde{\mathcal{M}}_H$ where additional states become massless at zero temperature is shown, provided the temperature is low enough compared to $M_{\rm min}$, the lower bound of the non-vanishing masses at P_0. Moreover, if at this point the h_{11} homology classes vanish, it follows that $M_{\rm min}$ must be of order $\mathcal{O}(e^{\phi}/\sqrt{\alpha'})$. Therefore, the massive contributions $\mathcal{O}(e^{-M_{\min}/T})$ we neglected in the free energy (see e.g. Eq. (2.23)) are exponentially suppressed, as soon as the universe exits the Hagedorn era and starts to cool.

At very early times, close to the Hagedorn temperature ($T \simeq e^{\phi}/\sqrt{\alpha'}$), the width of the potential $\mathcal{F}$ as a function of the moduli is very large. This follows from the fact that $e^{-M_s/T}$ (see Eq. (2.26)), where M_s is a moduli-dependent mass that vanishes at P_0, is not narrow when T is large. Therefore, even if initially the moduli fields sit at a point P very far from the local minimum at P_0, the well of the potential may contain both P and P_0, so that the system is dynamically attracted to a neighborhood of P_0, where the analysis of the present work starts to apply. Actually, the well of the effective potential may overlap many CY moduli spaces, such as $M, M', M'', \dots$ connected by extremal transitions, so that the dynamical mechanism of moduli stabilization may favor CY compactifications with large Hodge numbers, for the local minima of $\mathcal{F}$ to be deep.

5 Summary and perspectives

In this paper, we address the question of moduli stabilization in the context of type II superstring theory compactified on CY threefolds, once finite temperature is switched on. Even if the worldsheet conformal field theory is interacting, finite temperature can be implemented at the string level by a free orbifold action on the Euclidean time circle. This setup leads to no-scale models [6], *i.e.* classical theories where supersymmetry is spontaneously broken in flat Minkowski space. Therefore, flat directions of the classical potential exist, which can be organized as product of special Kähler and quaternionic manifolds, as follows from $\mathcal{N} = 2$ supersymmetry.

The above moduli spaces admit particular loci, where the internal manifold develops singularities when 2-cycles or 3-cycles collapse, implying generically massive supermultiplets to become massless. For instance in type IIA, BPS D2-branes on vanishing 2-cycles lead to hypermultiplets charged under $U(1)$ factors at conifold loci [12], or $SU(N)$ enhanced gauge symmetries coupled to g hypermultiplets at some "non-Abelian loci" [13]. We show that at least in the weak coupling regime, quantum/thermal effects stabilize the moduli at such particular points. The analysis is based on the one-loop low energy effective action, *without integrating out* the above additional light states in the sense of Wilsonian effective action, in order to avoid any IR divergence. We first determine the classical part of the action, which is a supergravity theory, whose gauging induces a potential we use to determine the moduli-dependent classical masses of the extra light states. At one-loop, the stringy Coleman-Weinberg effective potential depends on the classical masses and can be shown to admit local minima precisely where the light fields become massless, when the temperature is low enough.

The scalars that are stabilized are those belonging to the vector multiplets and hypermultiplets involved in the gauge theories geometrically engineered in the vicinities of the loci where the internal CY is singular. From the perspective of their string realization, they can either be non-perturbative fields, or perturbative one, in which case they are identified with flat directions of the initial classical potential. Therefore, the mechanism stabilizes both Kähler and complex structure moduli. In fact, the points in moduli space that are favored are situated at the intersection of several loci, each of which being associated to singularities developed by the internal space. We have argued that in general, say in type IIA, the entire Kähler moduli space is expected to be lifted, as well as the complex structure moduli associated to 3-cycles which can be resolved into 2-cycles.

In this setup, the temperature T is actually the no-scale modulus, also lifted by the thermal effective potential. However, instead of being stabilized (!) it acquires a run away behavior, which from a cosmological point of view arises when the flat universe expands. In other words, the model being non-supersymetric, time-translation is broken and the non-trivial one-loop contribution to the vacuum energy back-reacts on the classically static universe, which enters in quasi-static cosmological evolution. Homogenous and isotropic radiation dominated eras exist, characterized by static moduli sitting at their minima. They are particular solutions among more general ones where the massive moduli oscillate with damping around their minima. However, their "masses" happen to be proportional to the temperature, which is itself time-dependent and decreasing. As analyzed in detail in Ref. [11], the energy density stored in their oscillations scales as T^4 rather than T^3, as is the case when scalars have constant masses. As a result, moduli never dominate over radiation and the cosmological moduli problem [17] does not occur.

At one time or another, switching on finite temperature in a theoretical setup is certainly relevant to account for certain phases of the cosmological evolution of the universe. We stress that in the context of type II compactifications on CY threefolds, the effects described in this work should not be omitted. In the same process, they lead to stabilizations of moduli and determine the gauge group of the theory. In particular, the non-Abelian factors arise precisely at the points of enhanced gauge theory the moduli are attracted to.

However, more work is required to extend our results to compactifications on generalized CY spaces [33], including fluxes, branes and/or orientifold projections, leading to $\mathcal{N} = 1$ backgrounds at finite temperature. Furthermore, for $\mathcal{N} = 1$ to remain broken when the temperature is low and recover an MSSM-like model, an additional source of spontaneous breaking of supersymmetry should be implemented, whose origin may be attributed to internal fluxes. It would be interesting to extend to this context the results of Refs [18, 26, 34] derived in orbifold models. In these works, it is shown that the time trajectories of the scale of spontaneous supersymmetry breaking $M(t)$ and temperature $T(t)$ are attracted to a particular solution, where they are proportional to the inverse scale factor, $M(t) \propto T(t) \propto 1/a(t)$. Therefore, as the universe expands and cools, the hierarchy $M \ll M_{\text{Planck}}$ is dynamically generated.

Moreover, in models where the light spectrum is realistic enough, the Higgs mechanism should take place when the temperature is about the electroweak scale M_{ew}, in order to not screen radiative corrections. In this case, the evolution of the moduli masses proportional to T and M should halt at about M_{ew}, the scale where M is stabilized by radiative corrections [35, 36]. It is only at this stage that questions about dark matter may be addressed in this setup.

Finally, toroidal type II compactifications in presence of "gravito-magnetic" fluxes lead to thermal models, free of Hagedorn-like divergences [37]. The induced cosmological evolutions include bouncing [38] or emerging universes [39], where no initial singularity is encountered, while remaining in a perturbative regime. Therefore, it would be interesting to see if gravito-magnetic fluxes can be implemented in (generalized) CY compactifications and possibly lead to a theoretical framework able to account for both very early and very late times cosmological eras.

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Appendix A: 't Hooft symbols

We collect in this Appendix the definitions and useful properties of the 't Hooft symbols. They are denoted $\eta^{xu}{}_v$ and $\bar\eta^{xu}{}_v$ ($x = 1,2,3$; $u, v = 1,2,3,4$), are antisymmetric in u, v, and satisfy

$$\eta^{xu}{}_v = \bar\eta^{xu}{}_v = \epsilon^{xu}{}_v \quad (u, v = 1,2,3) \ , \qquad \eta^{xu}{}_4 = -\bar\eta^{xu}{}_4 = \delta^{xu}, \tag{A.1}$$

where $\epsilon^{123} = 1$. The indices u, v are equally up or down, since they are raised or lowered by Kronecker symbols. In matrix form, the 't Hooft symbols are written as

$$\eta^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} , \ \eta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \ \eta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} ,$$
$$\bar\eta^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \ \bar\eta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \ \bar\eta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{A.2}$$

and fulfill the relations

$$\eta^x \eta^y = -\delta^{xy} I_4 - \epsilon^{xyz} \eta^z \ , \qquad \bar\eta^x \bar\eta^y = -\delta^{xy} I_4 - \epsilon^{xyz} \bar\eta^z, \tag{A.3}$$

which imply

$$\mathrm{tr}\, \eta^x \eta^y = \mathrm{tr}\, \bar\eta^x \bar\eta^y = -4\delta^{xy}. \tag{A.4}$$

Summing over x, they give

$$\eta^{xt}{}_u \eta^{xv}{}_w = \delta^{tv}\delta_{uw} - \delta^t_w \delta^v_u + \epsilon^t{}_u{}^v{}_w \ , \qquad \bar\eta^{xt}{}_u \bar\eta^{xv}{}_w = \delta^{tv}\delta_{uw} - \delta^t_w \delta^v_u - \epsilon^t{}_u{}^v{}_w \ , \tag{A.5}$$

where $\epsilon^{1234} = 1$.

Appendix B: Canonical basis in hypergeometry

We first recall that the complex structures, metric and hyper-Kähler forms defined on a vectorial space take simple forms when they are written in a canonical base. We then apply these properties in the tangent plane at a given point P_0 of a quaternionic (or hyper-Kähler) manifold. Finally, we find the canonical form of the Killing vectors associated to Abelian isometries that fix P_0.

Theorem 1 : Let V be a $4n$-dimensional real vector space supplied with a triplet of complex structures J^x $(x = 1, 2, 3)$, satisfying the quaternionic algebra

$$J^x J^y = -\delta^{xy} I_{4n} + \epsilon^{xyz} J^z. \tag{B.1}$$

Let V^* be the dual of V. Then, there always exists some basis $e_{\mathcal{A}u}$ in V and its dual $\theta^{\mathcal{A}u}$ in V^*, where $\mathcal{A} = 1, \ldots, n$ and $u = 1, 2, 3, 4$, such that the complex structures take the following form, in terms of 't Hooft symbols:

$$J^x = -\delta^{\mathcal{A}}_{\mathcal{B}}\, \eta^{xu}{}_v\, e_{\mathcal{A}u} \otimes \theta^{\mathcal{B}v}. \tag{B.2}$$

Moreover, suppose V is endowed with a metric h, which is Hermitian under the three complex structures J^x $(x = 1, 2, 3)$,

$$\forall v, w \in V : \quad h(J^x v, J^x w) \equiv h(v, w), \tag{B.3}$$

and define the triplet of hyper-Kähler 2-forms K^x by

$$\forall v, w \in V : \quad K^x(v, w) \equiv h(J^x v, w). \tag{B.4}$$

Then, the basis $\{e_{\mathcal{A}u}\}$ can always be chosen orthonormal,

$$h = \delta_{\mathcal{AB}}\, \delta_{uv}\, \theta^{\mathcal{A}u} \otimes \theta^{\mathcal{B}v}, \tag{B.5}$$

and the hyper-Kähler 2-forms take the canonical form

$$K^x = \frac{1}{2}\, \delta_{\mathcal{AB}}\, \eta^x_{uv}\, \theta^{\mathcal{A}u} \wedge \theta^{\mathcal{B}v}. \tag{B.6}$$

Proof : Pick up any non-vanishing vector $e_{14} \in V$, and define $e_{1x} = -J^x e_{14}$. It is straightforward to consider an arbitrary linear combination of these four vectors to show that they are linearly independent. Denote $V_1 = \mathrm{Span}(e_{11}, e_{12}, e_{13}, e_{14})$ and repeat the previous steps by taking a non-vanishing $e_{24} \in V \backslash V_1$. Apply $-J^x$ on it to define e_{2x}, and hence $V_2 = \mathrm{Span}(e_{21}, e_{22}, e_{23}, e_{24})$. After repeating this procedure n times, we obtain $V = V_1 \oplus \cdots \oplus V_n$. It is easy to check that in the basis $e_{\mathcal{A}u}$ ($\mathcal{A} = 1, \ldots, n$; $u = 1, 2, 3, 4$), the Kähler forms J^x take the canonical form (B.2).

Next, defining $\theta^{\mathcal{A}u}$ to be the dual basis of V^*, we write the metric on V as

$$h = h_{\mathcal{A}u,\mathcal{B}v}\, \theta^{\mathcal{A}u} \otimes \theta^{\mathcal{B}v}, \tag{B.7}$$

and introduce the alternative notation $h^{(\mathcal{AB})}_{uv} = h_{\mathcal{A}u,\mathcal{B}v}$. Then, the Hermitian conditions (B.3) lead to

$$[\eta^x, h^{(\mathcal{AB})}] = 0. \tag{B.8}$$

This implies the real 4×4 matrix $h^{(\mathcal{AB})}$ can be written as

$$h^{(\mathcal{AB})} = h^{(\mathcal{BA})T} = \begin{pmatrix} a^{(\mathcal{AB})} & b^{(\mathcal{AB})} & c^{(\mathcal{AB})} & d^{(\mathcal{AB})} \\ -b^{(\mathcal{AB})} & a^{(\mathcal{AB})} & -d^{(\mathcal{AB})} & c^{(\mathcal{AB})} \\ -c^{(\mathcal{AB})} & d^{(\mathcal{AB})} & a^{(\mathcal{AB})} & -b^{(\mathcal{AB})} \\ -d^{(\mathcal{AB})} & -c^{(\mathcal{AB})} & b^{(\mathcal{AB})} & a^{(\mathcal{AB})} \end{pmatrix} \tag{B.9}$$

and satisfies

$$h^{(\mathcal{AB})} h^{(\mathcal{BA})} = \lambda^{(\mathcal{AB})}\, I_4 \quad \text{where} \quad \lambda^{(\mathcal{AB})} = (a^{(\mathcal{AB})})^2 + (b^{(\mathcal{AB})})^2 + (c^{(\mathcal{AB})})^2 + (d^{(\mathcal{AB})})^2. \tag{B.10}$$

In particular, $h^{(\mathcal{AA})}$ is diagonal with $a^{(\mathcal{AA})} > 0$, for the metric h to be definite positive. Thus, we can always rescale $e_{\mathcal{A}u}$ to effectively set $h^{(\mathcal{AA})} = I_4$. Clearly, such a rescaling does not spoil Eq. (B.2). Now, we introduce a scheme that removes the off-diagonal blocks of the metric, $h^{(\mathcal{AB})}$ for $\mathcal{A} \neq \mathcal{B}$, while keeping the standard form of the complex structures. We work this out block by block.

Take $(\mathcal{A}, \mathcal{B}) = (1, 2)$, and exhibit the relevant part of the metric as

$$h = (\theta^{1T}, \theta^{2T}) \begin{pmatrix} I_4 & h^{(12)} \\ h^{(21)} & I_4 \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} + \cdots . \tag{B.11}$$

It is straightforward to see that under the change of dual basis of V^* and V,

$$\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = \begin{pmatrix} I_4 & -h^{(12)} \\ 0_4 & I_4 \end{pmatrix} \begin{pmatrix} \theta'^1 \\ \theta'^2 \end{pmatrix}, \quad (e_1, e_2) = (e'_1, e'_2) \begin{pmatrix} I_4 & h^{(12)} \\ 0_4 & I_4 \end{pmatrix}, \tag{B.12}$$

the metric becomes

$$h = (\theta'^{1T}, \theta'^{2T}) \begin{pmatrix} I_4 & 0_4 \\ 0_4 & (1-\lambda^{(12)}) I_4 \end{pmatrix} \begin{pmatrix} \theta'^1 \\ \theta'^2 \end{pmatrix} + \cdots, \tag{B.13}$$

while the canonical form of the complex structures (B.2) is conserved, as follows from Eq. (B.8). Since Eq. (B.13) and the positive-definiteness of h imply $\lambda^{(12)} < 1$, it is possible to rescale the θ'^{2u}'s to absorb the factor $(1-\lambda^{(12)})$ in this equation. Again, this operation conserves the form of the complex structures.

Then, we apply the same procedure to eliminate the block $(\mathcal{A}, \mathcal{B}) = (1,3)$. It is easily seen that this procedure does not reintroduce a non-trivial bloc $(1,2)$. In general, one can show by double recursion that for $\mathcal{A} = 1, \ldots, n-1$ and $\mathcal{B} = \mathcal{A}+1, \ldots, n$, one can get rid off the blocks $h^{(\mathcal{A}\mathcal{B})}$. At the end of this process, the metric is diagonalized, $h = \theta^{\mathcal{A}u}\theta^{\mathcal{A}u}$, and Eq. (B.2) is valid. In this basis, the components of the hyper-Kähler forms are

$$K^x_{\mathcal{A}u,\mathcal{B}v} = h_{\mathcal{B}v,\mathcal{C}w} J^{x\mathcal{C}w}{}_{\mathcal{A}u} = \delta_{\mathcal{A}\mathcal{B}}\, \eta^x_{uv}. \tag{B.14}$$

Theorem 2 : Let $\mathcal{M}$ be a quaternionic (or hyperKähler) manifold of dimension $4n$, and any given point $P_0 \in \mathcal{M}$. Then, there exists some local coordinates $q^{\mathcal{A}u}$ ($\mathcal{A} = 1, \ldots, n; u = 1,2,3,4$) such that $q^{\mathcal{A}u}\big|_{P_0} = 0$ and the complex structures J^x, the metric h and the hyper-Kähler forms K^x at P_0 are:

$$J^x|_{P_0} = -\eta^{xu}{}_v \left(\frac{\partial}{\partial q^{\mathcal{A}u}} \otimes dq^{\mathcal{A}v} \right)\Big|_{P_0}, \tag{B.15}$$

$$h|_{P_0} = \left(dq^{\mathcal{A}u} dq^{\mathcal{A}u} \right)\big|_{P_0}, \tag{B.16}$$

$$K^x|_{P_0} = \frac{1}{2}\eta^x_{uv} \left(dq^{\mathcal{A}u} \wedge dq^{\mathcal{A}v} \right)\big|_{P_0}. \tag{B.17}$$

Proof : Consider a chart $\{\mathcal{U}, q^\Lambda\}$, where $\mathcal{U}$ is an open neighborhood of P_0 and q^Λ some coordinate system in $\mathcal{U}$. Let q_0^Λ be the coordinates of P_0. Applying Theorem 1 to the tangent plane at any point $P \in \mathcal{U}$, there exists a vielbein $\theta^{\mathcal{A}u}$ and its dual $e_{\mathcal{A}u}$ in $\mathcal{U}$, such that

$$J^x = -\eta^{xu}{}_v\, e_{\mathcal{A}u} \otimes \theta^{\mathcal{A}v}, \quad h = \theta^{\mathcal{A}u} \otimes \theta^{\mathcal{A}u}, \quad K^x = \frac{1}{2}\eta^x_{uv}\, \theta^{\mathcal{A}u} \wedge \theta^{\mathcal{A}v}. \tag{B.18}$$

We can write $\theta^{\mathcal{A}u} = U^{\mathcal{A}u}{}_\Lambda\, dq^\Lambda$ and $e_{\mathcal{A}u} = U^{-1\Lambda}{}_{\mathcal{A}u} \frac{\partial}{\partial q^\Lambda}$, where the matrix $\left(U^{\mathcal{A}u}{}_\Lambda\right)$ is invertible and depends smoothly on $P \in \mathcal{U}$. The new coordinates in $\mathcal{U}$

$$q^{\mathcal{A}u} := U^{\mathcal{A}u}{}_\Lambda\big|_{P_0} (q^\Lambda - q_0^\Lambda) \tag{B.19}$$

satisfy Eqs (B.15)–(B.17) and vanish at P_0.

Abelian isometries : In order to describe the charged hypermultiplets sector of an Abelian gauge theory, we suppose from now on the manifold $\mathcal{M}$ in Theorem 2 admits $U(1)^S$ isometries with fixed point P_0. Our aim is to find a canonical form for the Killing vectors at P_0.

We know from the first part of Theorem 1 (see Eqs (B.1) and (B.2)) applied to the tangent plane at P_0 that there is a system of coordinates $q^{\mathcal{A}u}$ in the neighborhood $\mathcal{U}$ of P_0 such that (B.15) is satisfied an $q^{\mathcal{A}u}\big|_{P_0} = 0$. We are interested in metrics on $\mathcal{M}$ admitting $U(1)^S$ isometries, whose Killing vectors have components $k_i^{\mathcal{A}u}$ ($i = 1, \dots, S$) admitting Taylor expansions of the form

$$k_i^{\mathcal{A}u} = Q_i^{\mathcal{A}u}\, t^u{}_v\, q^{\mathcal{A}u} + \mathcal{O}(q^2). \tag{B.20}$$

By construction, P_0 is fixed under the action of $U(1)^S$. Moreover, the isometries do not mix the components of different hypermultiplets. In other words, the quadruplet $(q^{\mathcal{A}1}, q^{\mathcal{A}2}, q^{\mathcal{A}3}, q^{\mathcal{A}4})$ has a well defined charge $Q_i^{\mathcal{A}}$ under the i^{th} $U(1)$. The $U(1)$ generator $t^u{}_v$ in Eq. (B.20) is determined by the convention to define complex numbers in the affine plane at P_0. For instance, the multiplication by the imaginary number i is represented by J^3, when we combine the $q^{\mathcal{A}u}$'s into complex numbers $q^{\mathcal{A}1} + iq^{\mathcal{A}2}$ and $q^{\mathcal{A}3} + iq^{\mathcal{A}4}$. In this case, the infinitesimal $U(1)^S$ transformations

$$\begin{aligned} e^{i\epsilon^i Q_i^{\mathcal{A}}}\,(q^{\mathcal{A}1} + iq^{\mathcal{A}2}) &= (q^{\mathcal{A}1} + iq^{\mathcal{A}2}) + \epsilon^i Q_i^{\mathcal{A}}(-q^{\mathcal{A}1} + iq^{\mathcal{A}1}) + \cdots\,, \\ e^{-i\epsilon^i Q_i^{\mathcal{A}}}\,(q^{\mathcal{A}3} + iq^{\mathcal{A}4}) &= (q^{\mathcal{A}3} + iq^{\mathcal{A}4}) + \epsilon^i Q_i^{\mathcal{A}}(q^{\mathcal{A}4} - iq^{\mathcal{A}3}) + \cdots\,, \end{aligned} \tag{B.21}$$

we want to represent with $\delta q^{\mathcal{A}u} = \epsilon^i k_i^{\mathcal{A}u}$ imply $t = -\bar{\eta}^3$.

A question then arises. Is the first order form of the Killing vector (B.20) conserved, when we diagonalize h, while keeping the canonical form of the complex structures J^x at P_0? The answer to this question is yes, due to the fact that the metric h must satisfy Killing equation

$$h_{\mathcal{A}u,\mathcal{C}w}\,\frac{\partial k_i^{\mathcal{C}w}}{\partial q^{\mathcal{B}v}} + h_{\mathcal{B}v,\mathcal{C}w}\,\frac{\partial k_i^{\mathcal{C}w}}{\partial q^{\mathcal{A}u}} = 0, \tag{B.22}$$

which at order zero in $q^{\mathcal{A}u}$ means

$$h_{\mathcal{A}u,\mathcal{B}w}\big|_{P_0}\, Q_i^{\mathcal{B}}\, t^u{}_v + h_{\mathcal{B}v,\mathcal{A}w}\big|_{P_0}\, Q_i^{\mathcal{A}}\, t^w{}_u = 0. \tag{B.23}$$

To diagonalize the hermitian metric h without spoiling the form of Eq. (B.15), we saw in the proof of Theorem 1 that we can set the blocks $h^{(\mathcal{AA})} = I_4$ and eliminate successively all non-diagonal 4×4 blocks $h^{(\mathcal{AB})}$ ($\mathcal{A} \neq \mathcal{B}$) of the metric by performing a sequence of changes of basis. These changes of bases for blocks $h^{(\mathcal{AA})}$ are only rescalings of the $q^{\mathcal{A}u}$'s which certainly conserve the form of the first order expansion of the Killing vectors $k_i^{\mathcal{C}w}\partial_{\mathcal{C}w}$. For the blocks $h^{(\mathcal{AB})}$ with $\mathcal{A} < \mathcal{B}$, they are of the form

$$\begin{pmatrix} q^{\mathcal{A}} \\ q^{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} I_4 & -h^{(\mathcal{AB})}\big|_{P_0} \\ 0_4 & I_4 \end{pmatrix} \begin{pmatrix} q'^{\mathcal{A}} \\ q'^{\mathcal{B}} \end{pmatrix}, \quad \left(\frac{\partial}{\partial q^{\mathcal{A}}}, \frac{\partial}{\partial q^{\mathcal{B}}}\right) = \left(\frac{\partial}{\partial q'^{\mathcal{A}}}, \frac{\partial}{\partial q'^{\mathcal{A}}}\right) \begin{pmatrix} I_4 & h^{(\mathcal{AB})}\big|_{P_0} \\ 0_4 & I_4 \end{pmatrix}, \tag{B.24}$$

which conserve the first order form of $k_i^{\mathcal{C}w}\partial_{\mathcal{C}w}$ as well, as can be checked using Eq. (B.23). To summarize, we have shown that there exists a system of coordinates $q^{\mathcal{A}u}$ on $\mathcal{M}$ such that $q^{\mathcal{A}u}\big|_{P_0} = 0$ and Eqs (B.15)–(B.17) and (B.20) are statisfied.

Appendix C

Theorem 3 : Let $\mathcal{M}$ be a quaternionic manifold of dimension $4n$ and ω^x the connection of the associated $SU(2)$ principal bundle. The fiber bundle over $\mathcal{M}$, whose fibers are triplets of $SU(2)$, does not admit non-trivial local parallel sections. In other words, the equation

$$\nabla^{SU(2)} L^x \equiv dL^x + \epsilon^{xyz}\omega^y L^z = 0 \tag{C.1}$$

in an open set $\mathcal{U}$ of $\mathcal{M}$ has only the solution $L^x = 0$.

Proof : We carry out a point-wise proof. For any given point $P_0 \in \mathcal{U}$, we consider the coordinate system $q^{\mathcal{A}u}$ of Theorem 2 and write Eq. (C.1) as

$$\forall P \in \mathcal{U}: \qquad \frac{\partial L^x}{\partial q^{\mathcal{A}u}} + \epsilon^{xyz}\omega^y_{\mathcal{A}u} L^z = 0. \tag{C.2}$$

Taking the partial derivative $\partial/\partial q^{\mathcal{B}v}$ and antisymmetrizing in $(\mathcal{A}u, \mathcal{B}v)$ yields

$$\epsilon^{xyz}\Omega^y_{\mathcal{A}u,\mathcal{B}v} L^z = 0, \tag{C.3}$$

where $\Omega^x = d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z$ is the curvature 2-form of the $SU(2)$-bundle. Since $\mathcal{M}$ is quaternionic, we have $\Omega^x = \lambda K^x$, where λ is a non-vanishing constant. Given the fact that $K^x|_{P_0}$ satisfies Eq. (B.17), we obtain $\epsilon^{xyz}\eta^y_{uv}\, L^z|_{P_0} = 0$. Multiplying with $\eta^{xw}{}_u$ and summing over x and u, Eq. (A.3) leads to $\eta^{zw}{}_v L^z|_{P_0} = 0$. Multiplying with $\eta^{xv}{}_w$ and summing over w and v, our desired result $L^x|_{P_0} = 0$ is obtained using Eq. (A.4).

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